# Accuracy of Simulations of the Gaussian random processes with continuous spectrum

**Yurii Kozachenko, Anatolii Pashko***

*Taras Shevchenko National University of Kyiv, Prospect Hlushkov, 4D, 03187 Kyiv, Ukraine*

*Received 1 March, 2014, www.tsi.lv*

**Abstract**

This paper investigates algorithms for the construction of sub-Gaussian models for the Gaussian stationary random processes with continuous spectrum. Estimates for random processes with analytical correlation functions retrieved and improved existing ones. Algorithms for simulation of random processes with given accuracy and reliability in various function spaces were constructed.

*Keywords*: Gaussian process, simulation, sub-Gaussian model, model accuracy, model reliability

## 1 Introduction

In this paper, we continue to examine algorithms for constructing sub-Gaussian models for Gaussian stationary random processes and fields [1-5]. To construct models of stochastic processes we use their spectral image in the form of stochastic integrals. In [6-9] we studied the modelling techniques and conditions of weak convergence models. Same works suggested to use the assessment points of difference for the process and model to assess the accuracy of the simulation. Papers [3-4] studied the rate of convergence of sub-Gaussian patterns in different functional spaces. It is possible to construct algorithms for simulation of random processes and fields with given accuracy and reliability for various functional spaces.

Strictly sub-Gaussian random spectral expansions of random processes was proposed to examine as the model. Sub-Gaussian properties of random processes researches are given in [10-11].

When constructing models of Gaussian random processes with given accuracy and reliability in various functional spaces we need to assess growth of $F(\infty)-F(\Lambda)\mathbb{F}$, where $\Lambda>0$, $F(\Lambda)$ - the spectral function of the process. If the spectral function is known, it is not difficult. Another situation where the only correlation function is known and the spectral function cannot be found explicitly.

Improved estimation for stationary Gaussian processes with continuous spectrum with correlation functions with desired properties was obtained in [1-5].

## 2 Basic concepts and definitions

Let $(\Omega,\mathrm{B},P)$ - be a standard probability space.

**Definition 1**. A random variable $\xi$ is called sub-Gaussian, if $E\xi=0$ and exists $a\geq 0$ such that $E\exp\{\lambda\xi\}\prec\exp\left\{\frac{\lambda^2a^2}{2}\right\}$ for all $\lambda\in R^1$.

A space of sub-Gaussian variables $Sub(\Omega)$ is Banach relative to the following norm $\tau(\xi)=\sup\limits_{\lambda\neq 0}\left[\frac{2\ln E\exp\{\lambda\xi\}}{\lambda^2}\right]^{\frac{1}{2}}$.

If $\tau(\xi)=E\xi^2$ - called strictly sub-Gaussian random variable.

Let $\xi(t)$ - be a real Gaussian stationary random process with - $E\xi(t)=0,\ R(\tau)-$ correlation function of $\xi(t)$, $F(\lambda)$ - the spectral function of the process $\xi(t)$, $R(\tau)=\int\limits_0^\infty\cos(\lambda t)dF(\lambda)$. A random process be represented as $\xi(t)=\int\limits_0^\infty\cos(\lambda t)d\xi_1(\lambda)+\int\limits_0^\infty\sin(\lambda t)d\xi_2(\lambda)$, where $\xi_1(t)$ and $\xi_2(t)$ - the centered and uncorrelated random processes with uncorrelated increments such as $0\leq\lambda_1<\lambda_2$ and $E\left(\xi_1(\lambda_2)-\xi_1(\lambda_1)\right)^2=$ $=E\left(\xi_2(\lambda_2)-\xi_2(\lambda_1)\right)^2=F(\lambda_2)-F(\lambda_1)$.

Let $D_\Lambda$ - be some partition of the interval $[0,\Lambda]$, $D_\Lambda:0=\lambda_0<\lambda_1<...<\lambda_n=\Lambda$. The model of random process $\xi(t)$ can be obtained as

$$S_n(t,\Lambda)=\sum_{i=0}^{n-1}\left[\cos(\lambda_i t)\left(\xi_1(\lambda_{i+1})-\xi_1(\lambda_i)\right)+\right.$$
$$\left.+\sin(\lambda_i t)\left(\xi_2(\lambda_{i+1})-\xi_2(\lambda_i)\right)\right].$$

* ***Corresponding author*** e-mail: pashkoua@mail.ru

The model of process $\xi(t)$ can be obtained by modelling the sum $\sum_{i=0}^{n-1}\left[\cos\left(\lambda_i t\right)\eta_{1i}+\sin\left(\lambda_i t\right)\eta_{2i}\right]$, where $\{\eta_{1i},\eta_{2i}\}$ - centered uncorrelated Gaussian random variables with $E\left(\eta_{1i}\right)^2=E\left(\eta_{2i}\right)^2=F(\lambda_{i+1})-F(\lambda_i)$. Given the precision of work with real numbers and algorithms errors for simulation Gaussian random variables, lets consider $\{\eta_{1i},\eta_{2i}\}$ - be a sequence of strictly sub-Gaussian uncorrelated random variables. Let $\xi(t)$ and all $S_n\left(t,\Lambda\right)$ belongs to certain functional Banach space $A(T)$ with norm of $\|\cdot\|$. Let the two numbers be as follow $\delta>0$ and $0<\alpha<1$.

**Definition 2**. Model $S_n\left(t,\Lambda\right)$ approximates the process $\xi(t)$ with reliability $1-\varepsilon$ and accuracy $\delta$ in the norm of space $A(T)$, if the following inequality holds $P\left\{\left\|\xi(t)-S_n\left(t,\Lambda\right)\right\|>\delta\right\}\le\varepsilon.$

## 3 Main results

**Theorem 1**. Model $S_n\left(t,\Lambda\right)$ approximates the process $\xi(t)$ with reliability $1-\varepsilon$ and accuracy $\delta$ in the norm of process $L_2(T)$, if for the numbers $\Lambda$ and $n$ the following inequalities hold $B_{n,\Lambda}<\delta^2$ and $\exp\left\{\frac{1}{2}\right\}\frac{\delta}{\sqrt{B_{n,\Lambda}}}\exp\left\{-\frac{\delta^2}{2B_{n,\Lambda}}\right\}\le\varepsilon$, where $B_{n,\Lambda}=\int_T E\left(\xi(t)-S_n(t,\Lambda)\right)^2 d\mu(t)$.

The proof of the theorem can be found in [3-5].

When for the correlation function $R(\tau)$ and spectral function $F(\lambda)$ of process $\xi(t)$ performs $R(0)=F\left(\infty\right)=1$ (in the general case we can assume $R(\tau)=\frac{R(\tau)}{R(0)}$ and $F(\lambda)=\frac{F(\lambda)}{F(\infty)}$), then $R(\tau)$ can be considered as the characteristic function of some random variable with symmetric distribution function $G(\lambda)$ and when $\lambda>0$ we have $F(\lambda)=G(\lambda)-G(-\lambda)$.

The following assertion holds.

**Theorem 2**. Let's assume that for the correlation function $R(\tau)$ and for some $t_0$ when $|t|<t_0$ the following condition holds $1-\frac{R(t)}{R\left(\infty\right)}\le\sum_{j=1}^{K}\psi_j(t)$, where $\psi_j(t),\ j=1,2...K$ - monotone nondecreasing functions, $\psi_j(0)=0$, such as, when $|uv|<t_0$, $\psi_j(uv)\le\psi_j^1(u)\psi_j^2(u)$, where $\psi_j^i(u)\ge0, i=1,2,\ j=1,2...K$ - monotone nondecreasing functions, $\psi_j^i(0)=0$, then following inequalities hold

1) if $h>\frac{2}{t_0}$, then $1-F(h)\le\frac{2}{\pi}\sum_{j=1}^{K}\mathrm{H}_{0,\psi_j}\left(\frac{h}{2}\right)$,

2) if $h>\frac{3}{t_0}$, then $1-F(h)\le\frac{8}{3\pi}\sum_{j=1}^{K}\mathrm{H}_{1,\psi_j}\left(\frac{h}{3}\right)$,

3) if $h>\frac{4}{t_0}$, then $1-F(h)\le\frac{3}{\pi}\sum_{j=1}^{K}\mathrm{H}_{2,\psi_j}\left(\frac{h}{4}\right)$,

where
$$\mathrm{H}_{n,\psi_j}\left(h\right)=\psi_j^1\left(\frac{1}{h}\right)\left(\int_0^1\psi_j^2(u)du+\int_1^{ht_0}\frac{\psi_j^2(u)}{u^{n+2}}du\right)+ +\frac{2}{(n+1)\left(ht_0\right)^{n+1}}.$$

The theorem is a general result of Lemma 4 and Theorem 1 from [2].

**Corollary 1**. Let assume that for the correlation function $R(\tau)$ and some $t_0$ when $|t|<t_0$ the following condition holds $1-\frac{R(t)}{R\left(\infty\right)}\le C_\alpha|t|^\alpha$, $0\le\alpha\le2$. Then for $1-\frac{F(\lambda)}{F(\infty)}$ we have the following estimates

1) if $h>\frac{2}{t_0}$, $0\le\alpha<1$,

$$\text{then } 1-\frac{F(h)}{F\left(\infty\right)}\le\frac{2}{\pi}\mathrm{H}'_{0,\alpha}\left(\frac{h}{2}\right), \quad (1)$$

2) if $h>\frac{3}{t_0}$, $0\le\alpha<2$,

$$\text{then } 1-\frac{F(h)}{F\left(\infty\right)}\le\frac{8}{3\pi}\mathrm{H}'_{1,\alpha}\left(\frac{h}{3}\right), \quad (2)$$

3) if $h>\frac{4}{t_0}$, $0\le\alpha\le2$,

$$\text{then } 1-\frac{F(h)}{F\left(\infty\right)}\le\frac{3}{\pi}\mathrm{H}'_{2,\alpha}\left(\frac{h}{4}\right), \quad (3)$$

where
$$H'_{n,\alpha}(h)=\frac{C_\alpha}{h^\alpha}\left(\frac{1}{\alpha+1}+\frac{1}{n+1-\alpha}\right)+ +\frac{1}{\left(ht_0\right)^{n+1}}\left(\frac{2}{n+1}-\frac{C_\alpha t_0^\alpha}{n+1-\alpha}\right).$$

The proof follows directly from Theorem 2.

These estimates depend on $\alpha$, so, having small $\alpha$ (1) is a better estimate, but when $\alpha>1$ then estimate (3) is better. Estimate (2) should be compared in each particular case. Obtained results can be used for evaluation $F(\infty)-F(\Lambda)$.

Suppose that the correlation function $R(\tau)$ has derivatives of order $4k, k=1,2,...$. Since

$R^{(4k)}(\tau)=(-1)^{2k}\int_0^\infty \lambda^{4k}\cos(\lambda\tau)dF(\lambda)$, then $R^{(4k)}(\tau)-$ the correlation function of a random process with a spectral function $F_k(\lambda)=\int_0^\lambda u^{4k}dF(u)$.

There is an inequality $F_k(\infty)-F_k(\Lambda)=\int_\Lambda^\infty u^{4k}dF(u)\ge\Lambda^{4k}\left(F(\infty)-F(\Lambda)\right)$, then $F(\infty)-F(\Lambda)\le\frac{1}{\Lambda^{4k}}\left(F_k(\infty)-F_k(\Lambda)\right)$.

**Corollary 2**. Suppose $k=1,2,...$ for the correlation function $R^{(4k)}(\tau)$ and for some $t_0$ when $|t|<t_0$ the following condition holds $1-\frac{R^{(4k)}(t)}{R^{(4k)}(\infty)}\le\sum_{j=1}^{K}\psi_{j,k}(t)$, where $\psi_{j,k}(t),\ j=1,2...K$ satisfies the conditions of the theorem 2. Then for $1-\frac{F_k(\lambda)}{F_k(\infty)}$ we have

1) if $h>\frac{2}{t_0}$, then $F_k(\infty)-F_k(h)\le\frac{2}{\pi}F_k(\infty)\sum_{j=1}^{K}\mathrm{H}_{0,\psi_j}\left(\frac{h}{2}\right)$,

2) if $h>\frac{3}{t_0}$, then $F_k(\infty)-F_k(h)\le\frac{8}{3\pi}F_k(\infty)\sum_{j=1}^{K}\mathrm{H}_{1,\psi_j}\left(\frac{h}{3}\right)$,

3) if $h>\frac{4}{t_0}$, then $F_k(\infty)-F_k(h)\le\frac{3}{\pi}F_k(\infty)\sum_{j=1}^{K}\mathrm{H}_{2,\psi_j}\left(\frac{h}{4}\right)$, where

$$\mathrm{H}_{n,\psi_j}(h)=\psi_j^1\left(\frac{1}{h}\right)\left(\int_0^1\psi_j^2(u)du+\int_1^{ht_0}\frac{\psi_j^2(u)}{u^{n+2}}du\right)+\frac{2}{(n+1)(ht_0)^{n+1}}.$$

The proof follows directly from Theorem 2.

More accurate estimates can be obtained when $R(\tau)$ is analytical. Let $\zeta$ be a positive probability unit, $F(x)$ - its distribution function and $\phi(\tau)=\int_0^\infty\cos(u\tau)dF(u)$ - the characteristic function.

**Lemma 1**. If $\phi(\tau)$ analytic in some district of zero and there are numbers $L>0$ and $\alpha>0$, such as for all $k\ge1$ the following condition holds $\left|\phi^{(2k)}(0)\right|\le(Lk)^{\alpha k}$, then in case of $h>(Le)^{\frac{\alpha}{2}}$ the following inequality holds $1-F(h)=P\{\zeta\ge h\}\le\exp\left\{-\frac{h^{\frac{2}{\alpha}}\alpha}{2Le}\right\}$.

**Proof**. From the Lyapunov inequality follows that when $2k-2\le s\le2k$, $k\ge1$ we have $\left(E\zeta^s\right)^{\frac{1}{s}}\le\left(E\zeta^{2k}\right)^{\frac{1}{2k}}$. Whereas, $E\zeta^{2k}=\left|\phi^{(2k)}(0)\right|=\int_0^\infty x^{2k}dF(x)$ then when $s\ge1$ there are relation

$$E\zeta^s\le\left(E\zeta^{2k}\right)^{\frac{1}{2k}}\le(Lk)^{\frac{\alpha s}{2}}\le(Ls)^{\frac{\alpha s}{2}}\left(\frac{k}{s}\right)^{\frac{\alpha s}{2}}\le(Ls)^{\frac{\alpha s}{2}}.$$

So, when $s\ge1$ we have $E\zeta^s\le(Ls)^{\frac{\alpha s}{2}}$. From this inequality and Chebyshev inequality implies that $h>0,s>1$ the following irregularities take place $P\{\zeta>h\}\le\frac{E\zeta^s}{h^s}\le\left(\frac{Ls}{h^{\frac{2}{\alpha}}}\right)^{\frac{\alpha s}{2}}$. Minimizing the right side by $s$, in this case $s=\frac{h^{\frac{2}{\alpha}}}{Le}$, we obtain the required inequality.

Let $F(u),u\ge0$ - be a spectral function of a stationary process. Since function $\frac{F(u)}{F(\infty)}$ can be considered as a function of the distribution of positive random variable $\zeta$, then we have the following theorem.

**Theorem 3**. Let the correlation function $R(\tau)$ of a stationary random process $\xi(t)$ be analytic in some district of zero and there are numbers $L>0$ and $m>0$, such as for all $k\ge1$ we have the following estimate $\left|R^{(2k)}(0)\right|\le(Lk)^{mk}$, then when $h>(Le)^{\frac{m}{2}}$ the following inequality holds $1-F(h)\le\exp\left\{-\frac{h^{\frac{2}{m}}m}{2Le}\right\}$.

## 4 Examples

Consider a stationary random process $\xi(t)$ with correlation function $R(\tau)=A\exp\{-C|\tau|\}$, where $A>0$, $C>0$. Spectral function $F(\lambda)=\frac{A}{\pi}arctg\left(\frac{\lambda}{C}\right)$ and $F(\infty)-F(\Lambda)=\frac{A}{\pi}\left(\frac{\pi}{2}-arctg\left(\frac{\Lambda}{C}\right)\right)$.

Let for stationary random process $\xi(t)$ correlation function $R(\tau)=A\exp\{-C|\tau|\}\cos(\beta\tau)$, where $A>0$,

$C>0$ and $\beta>0$. Spectral function $f(\lambda)=\frac{A}{\pi}\frac{C(C^2+\beta^2+\lambda^2)}{(\lambda^2-\beta^2-C^2)^2+4C^2\lambda^2}$. For $F(\infty)-F(\Lambda)$ the following holds

$$F(\infty)-F(\Lambda)=$$

$$=\frac{AC}{2\pi}\int_{\Lambda}^{\infty}\left(\frac{1}{C^2+(\beta+x)^2}+\frac{1}{C^2+(\beta-x)^2}\right)dx=.$$

$$=\frac{A}{2\pi}\left(\pi-arctg\left(\frac{\Lambda+\beta}{C}\right)-arctg\left(\frac{\Lambda-\beta}{C}\right)\right)$$

Consider a stationary random process $\xi(t)$ with correlation function $R(\tau)=A_\alpha\exp\left\{-C|\tau|^\alpha\right\}\cos(\beta\tau)$, where $A_\alpha>0$, $C>0$, $\beta>0$ and $1<\alpha\le2$. Let us estimate $F(\infty)-F(\Lambda)$ for the corresponding spectral function. Set $R(0)=F(\infty)=A_\alpha=1$.

Let $1<\alpha<2$, since when $x>0$ the following inequalities $\exp\{-x\}\ge1-x$ and $\cos(x)\ge1-\frac{x^2}{2}$, and when $|\tau|\le t_0=\min\left(C^{-\frac{1}{\alpha}},\sqrt{2}\beta^{-1}\right)$ we have

$$1-R(\tau)\le1-\left(1-C|\tau|^\alpha\right)\left(1-\frac{\beta^2\tau^2}{2}\right)=C|\tau|^\alpha+$$

$$+\frac{\beta^2\tau^2}{2}-C|\tau|^\alpha\left(\frac{\beta^2\tau^2}{2}\right)\le C|\tau|^\alpha+\frac{\beta^2\tau^2}{2}.$$

Thus, according to corollary 1, when $h>\frac{4}{t_0}$ we have $1-F(h)\le\frac{3}{\pi}\left(H'_{2,\alpha}\left(\frac{h}{4}\right)+H'_{2,2}\left(\frac{h}{4}\right)\right)$, where $H'_{2,2}(a)=\frac{3\beta^2}{4a^2}$ and $H'_{2,\alpha}(a)=\frac{C}{a^\alpha}\left(\frac{1}{\alpha+1}+\frac{1}{3-\alpha}\right)$.

That is, when $h>\frac{4}{t_0}$ we have

$$F(\infty)-F(h)\le\frac{3A_\alpha}{\pi}\left(H'_{2,\alpha}\left(\frac{h}{4}\right)+H'_{2,2}\left(\frac{h}{4}\right)\right)=$$

$$=\frac{3A_\alpha}{\pi}\left(\frac{4^\alpha C}{h^\alpha}\left(\frac{1}{\alpha+1}+\frac{1}{3-\alpha}\right)+\frac{12\beta^2}{h^2}\right).$$

Let $\alpha=2$. Note that $R(\tau)=\exp\left\{-C|\tau|^2\right\}\cos(\beta\tau)$ - is the product of two characteristic functions: $\psi_1(\tau)=\exp\left\{-C|\tau|^2\right\}$ - characteristic function of a random variable $\zeta_1$, which has a normal distribution from $E\zeta_1=0$, $E\zeta_1^2=\frac{1}{2C}$, and $\psi_2(\tau)=\cos(\beta\tau)$ - characteristic function of a random variable $\zeta_2$ with a distribution law $P\{\zeta_2=\beta\}=\frac{1}{2}$ and $P\{\zeta_2=-\beta\}=\frac{1}{2}$.

So $R(\tau)$ – is the characteristic function of a random variable $\zeta=\zeta_1+\zeta_2$, where $\zeta_1$ and $\zeta_2$ are independent. That is, if $\lambda>0$

$$1-F(h)=P\{|\zeta|>h\}=P\{|\zeta_1+\zeta_2|>h\}=$$

$$=\frac{P\{|\zeta_1+\beta|>h\}}{2}+\frac{P\{|\zeta_1-\beta|>h\}}{2}.$$

If $h>\beta$, it is easy to see

$$1-F(h)=\frac{1}{2}\left(P\{|\zeta_1|>h-\beta\}+P\{|\zeta_1|>h+\beta\}\right)=$$

$$=\left(\frac{C}{\pi}\right)^{\frac{1}{2}}\left(\int_{\lambda-h}^{\infty}\exp\{-t^2C\}dt+\int_{\lambda+h}^{\infty}\exp\{-t^2C\}dt\right)=$$

$$=\left(\frac{1}{\pi}\right)^{\frac{1}{2}}\left(\int_{(h-\beta)\sqrt{C}}^{\infty}\exp\{-u^2\}du+\int_{(h+\beta)\sqrt{C}}^{\infty}\exp\{-u^2\}du\right).$$

So,

$$1-F(h)\le\frac{1}{\sqrt{\pi}}\left(\frac{2}{(h-\beta)\sqrt{C}}\exp\{-C(h-\beta)^2\}+\right.$$

$$\left.+\frac{2}{(h+\beta)\sqrt{C}}\exp\{-C(h+\beta)^2\}\right).$$

That is, when $h>\beta$ we have the following inequality

$$F(\infty)-F(h)\le\frac{2A_2}{\sqrt{\pi C}}\left(\frac{2}{(h-\beta)\sqrt{C}}\exp\{-C(h-\beta)^2\}+\right.$$

$$\left.+\frac{2}{(h+\beta)\sqrt{C}}\exp\{-C(h+\beta)^2\}\right).$$

## 5 Estimation of model parameters

Let the random process $\xi(t)$ be defined on the interval $[0,T], T>0$. Model of random process $\xi(t)$ built as $S_n(t,\Lambda)=\sum_{i=0}^{n-1}\left[\cos(\lambda_i t)\eta_{1i}+\sin(\lambda_i t)\eta_{2i}\right]$, where $D_\Lambda:0=\lambda_0<\lambda_1<...<\lambda_n=\Lambda$ - uniform partition of the interval $[0,\Lambda]$, and $\{\eta_{1i},\eta_{2i}\}$ - strictly sub-Gaussian independent random variables with $E(\eta_{1i})^2=E(\eta_{2i})^2=F(\lambda_{i+1})-F(\lambda_i)$.

Let's find such $\Lambda$ and $M$, that the selected model approximates the process $\xi(t)$ with given accuracy $\delta$ and reliability $1-\varepsilon$ in $L_2(T)$.

By the theorem 1 for the unknown $\Lambda$ and $M$ the following inequalities must hold $B_{M,\Lambda}\le\delta^2$ and

$B_{M,\Lambda} \le \frac{\delta^2}{s_\varepsilon}$, where $s_\varepsilon$ - the root of the equation $\exp\left\{\frac{1}{2}-\frac{s^2}{2}\right\}s=\varepsilon$.

For random process $\xi(t)$ with correlation function $R(\tau)=A\exp\{-C|\tau|\}$, where $A>0$, $C>0$ and uniform partition of the interval $[0,\Lambda]$ the following holds $B_{n,\Lambda} \le \frac{T^3\Lambda^2}{3M^2}F(\Lambda)+T(F(\infty)-F(\Lambda))$. Hence, $B_{M,\Lambda} \le \frac{T^3\Lambda^2}{3M^2}F(\Lambda)+\frac{TA}{\pi}\left(\frac{\pi}{2}-arctg\left(\frac{\Lambda}{C}\right)\right)$ and $E(\eta_{1i})^2=E(\eta_{2i})^2=\frac{A}{\pi}\left(arctg\left(\frac{\lambda_{i+1}}{C}\right)-arctg\left(\frac{\lambda_i}{C}\right)\right)$.

For random process $\xi(t)$ with correlation function $R(\tau)=A\exp\{-C|\tau|\}\cos(\beta\tau)$, where $A>0$, $C>0$, $\beta>0$ and uniform partition of the interval $[0,\Lambda]$ the following holds

$$B_{M,\Lambda} \le \frac{T^3\Lambda^2}{3M^2}F(\Lambda)+\frac{TA}{2\pi}\left(\pi-arctg\left(\frac{\Lambda+\beta}{C}\right)-arctg\left(\frac{\Lambda-\beta}{C}\right)\right)$$

and $E(\eta_{1i})^2=E(\eta_{2i})^2=\frac{A}{2\pi}\left(arctg\left(\frac{\lambda_{i+1}+\beta}{C}\right)+\right.$

$$\left.+arctg\left(\frac{\lambda_{i+1}-\beta}{C}\right)-arctg\left(\frac{\lambda_i+\beta}{C}\right)-arctg\left(\frac{\lambda_i-\beta}{C}\right)\right).$$

For stationary random process $\xi(t)$ with correlation function $R(\tau)=A_\alpha\exp\left\{-C|\tau|^\alpha\right\}\cos(\beta\tau)$, where $A_\alpha>0$, $C>0$, $\beta>0$, $1<\alpha\le 2$ and

$$E(\eta_{li})^2=F(u_{i+1})-F(u_i)=\frac{2}{\pi}\int_{u_i}^{u_{i+1}}\int_0^\infty\cos(v\tau)R(\tau)d\tau dv=$$

$$=\frac{2A_\alpha}{\pi}\int_0^\infty\left(\sin\left(\frac{(i+1)\Lambda\tau}{M}\right)-\sin\left(\frac{i\Lambda\tau}{M}\right)\right)\times \exp\left\{-C|\tau|^\alpha\right\}\frac{\cos(\beta\tau)}{\tau}d\tau.$$

Consider $\Lambda$, when $\Lambda\ge\Lambda_0=\max\left(1,\frac{1}{t_0}\right)$, where $t_0=\min\left(C^{-\frac{1}{\alpha}},\sqrt{2}\beta^{-1}\right)$. Then we have $F(\infty)-F(\Lambda)\le\frac{A_\alpha L_\alpha}{\Lambda^\alpha}$, where $L_\alpha=\frac{3}{\pi}\left(\frac{C4^{\alpha+1}}{(\alpha+1)(3-\alpha)}+12\beta^2\right)$.

Therefore, $B_{M,\Lambda}\le\frac{T^3\Lambda^2}{3M^2}F(\infty)+\frac{TA_\alpha L_\alpha}{\Lambda^\alpha}$. Minimizing the right side by $\Lambda$, so $\Lambda_M=\left(\frac{3L_\alpha\alpha M^2}{2T^2}\right)^{\frac{1}{2+\alpha}}$, we get the inequality

$$B_{M,\Lambda}\le\frac{T^{\frac{3\alpha+2}{\alpha+2}}}{M^{\frac{2\alpha}{2+\alpha}}}\left(\frac{2A_\alpha}{3}\right)^{\frac{\alpha}{\alpha+2}}(L_\alpha A_\alpha\alpha)^{\frac{2}{2+\alpha}}(\alpha+2).$$

So, for $M$ the inequality must holds

$$M\ge\frac{T^{\frac{3\alpha+2}{\alpha+2}}A_\alpha^{\frac{1}{2}+\frac{1}{\alpha}}\left(\frac{2}{3}\right)^{\frac{1}{2}}(L_\alpha\alpha)^{\frac{1}{\alpha}}(\alpha+2)^{\frac{\alpha+2}{2\alpha}}}{\left(\delta^2\min(1,s_\varepsilon^{-1})\right)^{\frac{\alpha+2}{2\alpha}}}=N_1.$$

**Lemma 2**. Model $S_n(t,\Lambda)$ approximates the process $\xi(t)$ with reliability $1-\varepsilon$ and accuracy $\delta$ in the norm $L_2([0,T])$, if $\Lambda=\Lambda_M$, and for $M$ the following inequality holds $M\ge\max(N_1,N_2,N_3)$, where

$$N_2=\Lambda_0^{\frac{\alpha+2}{2}}\left(\frac{2}{3L_\alpha\alpha}\right)^{\frac{1}{2}}T,\qquad N_3=\left(\frac{3}{2}L_\alpha\alpha\right)^{\frac{1}{\alpha}}T,$$

$$\Lambda_0=\max\left(1,\frac{1}{t_0}\right),\ t_0=\min\left(C^{-\frac{1}{\alpha}},\sqrt{2}\beta^{-1}\right).$$

Same results can be achieved in the space $L_p([0,T]),\ p>2$.

**Lemma 3.** Model $S_n(t,\Lambda)$ approximates the process $\xi(t)$ with reliability $1-\varepsilon$ and accuracy $\delta$ in the norm $L_p([0,T]),\ p>2$, if $\Lambda=\Lambda_M$, and for $M$ the following inequality holds $M\ge\max(N_4,N_5)$, where

$$N_4\frac{\sqrt{2}T^{\frac{\alpha+2+p\alpha}{\alpha p}}A_\alpha^{\frac{1}{2}+\frac{1}{\alpha}}(L_\alpha\alpha)^{\frac{1}{\alpha}}(\alpha+2)^{\frac{\alpha+2}{2\alpha}}}{\left(\delta^2\min\left((p+1)^{-1},s_\varepsilon^{-2}\right)\right)^{\frac{\alpha+2}{2\alpha}}},$$

$$N_5=\Lambda_0^{\frac{\alpha+2}{2}}\left(\frac{2}{L_\alpha\alpha}\right)^{\frac{1}{2}}T.$$

Proof. Let $\mu(T)=T$. For the unknown $\Lambda$ and $M$ the following inequalities must hold $T^{\frac{2}{p}}D_{M,\Lambda}\le\frac{\delta^2}{s_\varepsilon^2}$ and $T^{\frac{2}{p}}D_{M,\Lambda}\le\frac{\delta^2}{p+1}$. Consider $\Lambda$, as $\Lambda\ge\Lambda_0=\max\left(1,\frac{1}{t_0}\right)$, the following inequality must hold $D_{M,\Lambda} \le \frac{T^2\Lambda^2}{M^2}F(\infty) + \frac{A_\alpha L_\alpha}{\Lambda^\alpha}$. Minimizing the right side by $\Lambda$, i.e., if put $\Lambda = \Lambda_M = \left(\frac{L_\alpha \alpha M^2}{2T^2}\right)^{\frac{1}{2+\alpha}}$, then we get inequality $D_{M,\Lambda} \le \frac{T^{\frac{\alpha+2}{2\alpha}}}{M^{\frac{2\alpha}{2+\alpha}}}(2A_\alpha)^{\frac{\alpha}{\alpha+2}}(L_\alpha A_\alpha \alpha)^{\frac{2}{2+\alpha}}(\alpha+2)$. So, for $M$ the following inequality must hold

$$M \ge \frac{\sqrt{2T}^{\frac{\alpha+2+p\alpha}{\alpha p}} A_\alpha^{\frac{1}{2}+\frac{1}{\alpha}} (L_\alpha \alpha)^{\frac{1}{\alpha}} (\alpha+2)^{\frac{\alpha+2}{2\alpha}}}{\left(\delta^2 \min\left((p+1)^{-1}, s_\alpha^{-2}\right)\right)^{\frac{\alpha+2}{2\alpha}}} = N_4 .$$

Given that $\Lambda_M \ge \Lambda_0$, we get the proof of the lemma.

In the real simulations, for example, for a given reliability $1-\varepsilon$ and accuracy $\delta$ at $A = A_\alpha = 1$, $C = 1$ and $T = 1$ we have the following results (table 1) in $L_2([0,T])$.

TABLE 1 Estimates for the parameters $M$ and $\Lambda$ for the given accuracy and reliability in $L_2([0,T])$

| $\delta$ | $\varepsilon$ | $\beta$ | $\alpha$ | $M$ | $\Lambda$ |
|---|---|---|---|---|---|
| 0.1 | 0.05 | 0 | 1 | 17000 | 400 |
| 0.05 | | | | 137000 | 1600 |
| 0.01 | | | | 20000000 | 35000 |
| 0.1 | 0.05 | 1 | 1 | 24000 | 420 |
| 0.05 | | | | 199000 | 150 |
| 0.1 | 0.05 | 1 | 1.1 | 175200 | 6925 |
| 0.05 | | | | 1236000 | 24420 |
| 0.01 | | | | 115300000 | 455500 |
| 0.1 | 0.05 | 1 | 1.5 | 27060 | 1010 |
| 0.05 | | | | 136400 | 2535 |
| 0.01 | | | | 5830000 | 21680 |
| 0.1 | 0.05 | 1 | 2 | 8080 | 281 |
| 0.05 | | | | 32320 | 562 |
| 0.01 | | | | 808000 | 2810 |

## 6 Conclusions

The paper shows estimates for increases of spectral functions of random processes. The results are used for the parameter estimations of sub-Gaussian models in the simulation of Gaussian random processes. Found assessment allow us to build a model with given accuracy and reliability in various function spaces, in particular $L_2([0,T])$, $L_p([0,T])$, $p > 2$.

## References


[1] Kozachenko Yu V, Pashko A A 1988 Simulation of Gaussian stationary stochastic processes represented by stochastic integrals *Theory and applications of statistical simulation* Novosibirsk 10-24

[2] Kozachenko Yu V, Kozachenko L F 1992 On the modelling of Gaussian stationary processes with absolutely continuous spectrum *Probability Theory and Math. Statistics* **47** 47-54

[3] Kozachenko Yu V, Pashko A A 1998 The accuracy of the simulation of random processes in the Orlicz spaces I *Probability Theory and Math. Statistics* **58** 75-90

[4] Kozachenko Yu V, Pashko A A 1998 The accuracy of the simulation of random processes in the Orlicz spaces II *Probability Theory and Math. Statistics* **59** 45-60

[5] Pashko A A 2006 Numerical simulation of sub-Gaussian random fields *Bulletin of Taras Shevchenko National University in Kyiv. Physics and Mathematics* **1** 35-39

[6] Ermakov S M, Mikhailov G A 1982 *Statistical simulations* Nauka: Moscow p 296

[7] Prigarin S M 1988 Simulation of Gaussian stationary stochastic processes represented by stochastic integrals *Theory and applications of statistical simulation* Novosibirsk 31-39

[8] Prigarin S M 2001 *Spectral Models of Random Fields in Monte Carlo Methods* Utrecht: VSP p 195

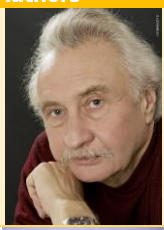
[9] Prigarin S M 2005 *Numerical Modelling of Random Processes and Fields* Novosibirsk p 259

[10] Buldygin V V, Kozachenko Yu V 2000 *Metric characterization of random variables and random processes* Amer. Math. Soc., Providence, RI p 260

[11] Kozachenko Yu V, Pashko A A 1999 *Simulation of random processes* Kyiv: T. Shevchenko University p 224



**Authors**



**Yurii Kozachenko, born on December 1, 1940, Kyiv, Ukraine**

**Current position, grades:** professor, doctor of sci. in physics and mathematics
**University studies:** Taras Shevchenko National University of Kyiv
**Scientific interest:** Analytical properties of stochastic processes, distribution estimation of functionals from random processes, random processes in Orlicz spaces, pre-Gaussian and sub-Gaussian random processes, Cauchy problem for mathematical physics equations with random initial conditions, simulation of random processes, statistics of random processes, wavelet expansions of random processes
**Publications:** 220

**Anatolii Pashko, born on February 28, 1962,Kuznetsovsk, Ukraine**

**Current position, grades:** assistant professor
**University studies:** Taras Shevchenko National University of Kyiv
**Scientific interest:** Stochastic processes theory, simulation of stochastic processes and fields
**Publications:** 100