Keywords: Metamorphic Mechanism; Screw Theory; Product of Exponentials Formula; Forward and Inverse Kinematics Solution

1 Introduction

The manufacturing of ship products is often accompanied by large-diameter hole cutting and processing on the hull surfaces, which are pieced together with complex surface patches in most cases. And then square groove or bevel preparation of these large size holes are often needed [1-2]. As a critical technical problem in shipbuilding industry, large-diameter-hole and changing-angle bevel structure processing is characterized by high difficulty, low efficiency and low accuracy, etc., and the processing quality of which will directly influence the quality and efficiency of the subsequent industrial processes.

Seamus Gordon, et al., from University of Limerick, designed a high-speed CNC cutting machine realizing spatial movement of the cutting tool along three axial directions. Controlled by computer numerical control programming, the high speed of this cutting device reaches 40m/min [3]. The Omnimat cutting machine developed and produced by Messer, is a high-performance, multi-functional, large CNC cutting machine applicable for high speed and precision cutting of large size flat plate [4]. The model RV-016 cutting robot, developed and produced by Panasonic Corporation, adopts a serial mechanism, and its end infinite rotating cutting gun can perform high speed and precision cutting task on complex surfaces.

In China, many experts and engineering technicians in machinery manufacturing field have made a series of researches on numerical control cutting technology. The multi-functional NC-fair incision machine with a gantry-bridge structure, designed by Ming, et al., can cut large 5000mm × 3000mm work-piece and perform arbitrary plane curve cutting tasks [5]. The cantilever structured CNC large intersecting circle flame cutting machine with five-bar cooperating, designed by Sun, et al., can process a work-piece with a diameter of 4000-8000mm (pipe); and the diameter of the cutting hole lies in a range of 50-550mm, and the bevel angle is between 30° and 40° [6]. Zhang, from Harbin Engineering University, designed a large complex surface cutting mechanism with the end cutting torch being a five-bar metamorphic mechanism to achieve square groove and bevel groove of large diameter hole cutting on complex surfaces; controlled by electric motor, it can perform spatial movements with changing pose and trajectory [7].

Based on the analysis of literatures at home and abroad, we know there are two types of cutting techniques and equipment: one is round hole cutting on a plate with its vertical displacement almost unchanged in the cutting process. And the bevel angle remain unchanged in bevel cutting; the other is CNC pipe-cutting machine, which is applicable for round pipe (whose diameter is less than 1.4m) intersection hole cutting.

The parallel mechanism has the merits of simple high-rigidity structure, high processing speed and accuracy, and is less affected by the inertia, etc. For these advantages over the serial mechanism, it has been widely employed in various processing and manufacturing industries. Furthermore, its end moving platform can many complex spatial motions via multi-axis coordination [8-13]. Studies about metamorphic mechanism starts late, but this mechanism has stimulated the research interests of many scholars for its flexible topology, based on which many tasks can be completed [14-21].

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According to the principles of metamorphic mechanism, a parallel cutting mechanism was designed to solve the problem of changing pose and trajectory in square groove and bevel groove cutting. Since there are different requirements for the cutting torch motion in the process of cutting different shapes of holes on complex surfaces, we reasonably designed different configuration frameworks of the metamorphic mechanism based on their metamorphic characteristics, which can meet the technology requirements of changing spatial pose and trajectory cutting on large complex surface.

2 Metamorphic Parallel Cutting Mechanism Design

2.1 A NOVEL DESIGN OF SPATIAL METAMORPHIC KINEMATIC PAIR

A conventional spherical pair has three spatial degrees of freedom (DOF). And there are 3 degrees of freedom of rotating between the two components linked by the spherical pair. In actual analysis, the motions of the two components can be simplified as the rotation around the three spatial axes that are perpendicular to each other and also meet the right-hand rule. Therefore, a spatial spherical pair is equivalent to a compound kinematic pair consisting of a Hooke’s joint and a revolute pair whose axis is perpendicular to the Hooke’s joint axial plane. For a Hooke’s joint, the shaft connected with the base is the main shaft, and that connected with the linkage is a counter shaft.

According to the principles of metamorphic mechanism, we designed a slot where the two ends of the counter shaft can slide on the main shaft of Hooke’s joint. The counter shaft can be fixed in a certain position as required with a pin; therefore, we can reduce the degree of freedom of the kinematic pair by putting pin into the pinhole located on the revolute pair. Based on the framework before and after metamorphic operation, the novel metamorphic kinematic pair can be classified into four configurations:

(1) The original configuration \( S_{m3} \)

Fix the counter shaft in a position perpendicular to the main shaft, and remove the pin on the revolute pair. The kinematic pair has three degrees of freedom and is equivalent to a spatial spherical pair, as shown in Figure 5(a);

(2) The a-type 2-DOF sub-configuration \( S_{n2}^a \)

Fix the counter shaft in a position perpendicular to the main shaft, and put the pin into the pin hole on the revolute pair. Now the kinematic pair has two degrees of freedom and is equivalent to a traditional Hooke’s joint, as shown in Figure 5(b);

(3) The b-type 2-DOF sub-configuration \( S_{n2}^b \)

Fix the counter shaft in a co-axial position with the main shaft, and remove the pin on the revolute pair. Now the kinematic pair has two degrees of freedom, as shown in Figure 5(c);

(4) Sub-configuration \( S_{m1} \) with single degree of freedom

Fix the counter shaft in a co-axial position with the main shaft, and remove the pin on the revolute pair. Now the kinematic pair has only one degree of freedom and is equivalent to a traditional revolute pair, as shown in Figure 1 (d);

2.2 THE METAMORPHIC PARALLEL CUTTING MECHANISM’S ORIGINAL CONFIGURATION FRAMEWORK

Based on the above designed metamorphic kinematic pair unit, a limb kinematic chain where we can switch the degree of freedom is constructed. Assign the position of each kinematic pair, and form the metamorphic parallel mechanism using three same limb chains (see Figure 2). The end effectors \( M_1M_2M_3 \) are connected with the fixed base \( B_1B_2B_3 \) via the three same kinematic chains \( S_{m1}S_{m2}^a S_{m2}^b J^1 \). Under this framework, it is 6-DOF mechanism that can achieve upper and lower bevel cutting. Therefore, it is regarded as the original configuration of the cutting machine. When the mechanical properties of the mechanism are considered, the driving input should be as near as possible to the fixed platform to reduce the influence of inertia and to improve load performance. So we choose three Hooke’s joints on the fixed platform as driving input.

2.3 THE NOVEL METAMORPHIC PARALLEL MECHANISM’S SUB-CONFIGURATION FRAMEWORK

Based on the original configuration, the 3-DOF sub-configuration framework of metamorphic parallel mechanism is obtained by transforming the metamorphic kinematic pair \( S_{m2}^a \) into \( S_{m1} \) framework, and the metamorphic kinematic pair \( S_{m2}^b \) into \( S_{m2}^b \). Then the limb chain structure changes after these metamorphic operations. The kinematic pair distribution in the limb chain turns into \( S_{m1}S_{m2}^a S_{m2}^b J^1 \), which means the sub-configuration framework of the metamorphic parallel cutting mechanism is \( 3S_{m1}S_{m2}^a S_{m2}^b J^1 \), as shown in Figure 3. With three revolute pairs on the fixed platform chosen
as driving inputs, the square groove cutting on the surface can be achieved.

The coordinates of point $M_i$ in the equation can be obtained through the following methods:

$$g_i(\theta) = e^{\omega_i e^{\alpha_i}} e^{\omega_i e^{\alpha_i}} g_i(0), \quad (2)$$

where

$$e^{\omega_i} = \begin{bmatrix} e^{\alpha_i} (I - e^{\alpha_i}) (e \times v) + \theta e e v \end{bmatrix} \quad (e \neq 0),$$

$$\xi = \begin{bmatrix} \omega \\ r \times \omega \end{bmatrix}, \quad \dot{\xi} = \begin{bmatrix} \dot{\omega} \\ r \times \dot{\omega} \end{bmatrix}, \quad \omega = \begin{bmatrix} \omega_i \\ \omega_j \end{bmatrix},$$

$$\omega_i = \begin{bmatrix} 0 & -\omega_j & \omega_k \\ -\omega_j & 0 & -\omega_k \\ -\omega_k & \omega_j & 0 \end{bmatrix}.$$
Then we have the equation, in which the highest exponent degree of \(x\) is not more than 2:

\[ F_{1x}x^2 + F_{2x}x + F_{3x} = 0, \]  

(9)

where \(F_i\) \((i = 1 \sim 3)\) are expressions in which the highest exponent degree of \(y\) is 2.

Eliminate the variable \(x\) using Sylvester resultant; regard \(x\) and the constant terms as new variables; multiply \(x\) with equation (5) and (10), combed with which a matrix is constructed. Since only non-zero solutions are considered, the resultant is zero:

\[ \begin{bmatrix} p_{1x} & p_{2x} & p_{3x} & 0 \\ 0 & p_{4x} & p_{5x} & p_{6x} \\ p_{4x} & p_{3x} & p_{6x} & 0 \\ 0 & p_{4x} & p_{5x} & p_{6x} \end{bmatrix} = 0. \]  

(10)

Then we have the equation in which the highest exponent degree of \(y\) is not more than 4:

\[ K_{1x}y^4 + K_{2y}y^3 + K_{3y}y^2 + K_{4y} + K_0 = 0, \]  

(11)

where \(K_{i}\) \((i = 0 \sim 4)\) is a polynomial containing \(\theta_{1i}, \theta_{2i}, \theta_{3i}\) \((i = 1 \sim 3)\).

Because the driving kinematic pair rotation angles \(\theta_{1i}, \theta_{2i}, \theta_{3i}\) are known quantities, the value of \(y\) can be calculated out from equation (11). Substituting the \(y\) value into equation (5) and (6), we yield two sets of values of \(x \) and \(z\), and then substituting, which into equation (7) to determine the end values of \(x\) and \(z\).

After that, the coordinates of \(M_1, M_2, M_3\) on the end moving platform can be determined, and then we can further determine the pose of the moving platform in the absolute coordinate system.

3.1.2 The analysis of the original configuration inverse kinematics solution

Supposing that the geometric center of the moving platform is \(C\), the cutting gun endpoint is \(P\), and the length of the cutting gun is \(n\), we set up the moving coordinate system Cxyz of the moving platform \(M_1M_2M_3\). The \(X\)-axis passes through point \(M_1\); \(Z\)-axis is perpendicular to the moving platform in an outward direction; and \(Y\)-axis is determined according to right-hand rule. The analysis of original configuration inverse kinematics solution is to determine the driving kinematic pair rotation angle \(\theta_{1i}\) and \(\theta_{2i}\) \((i = 1 \sim 3)\), based on the known pose of the end moving platform \(M_1M_2M_3\).

Supposing that the moving coordinate system Cxyz is formed by rotating the absolute coordinate system along the \(X\)-axis, \(Y\)-axis and \(Z\)-axis by the angles of \(\gamma, \beta, \alpha\) respectively, the coordinates of \(C\) are \((x_c, y_c, z_c)\), and the coordinates of \(P\) is \((x_P, y_P, z_P)\), the pose of centre of mass (PCM) of the end moving platform is represented as:

\[ g_c = \begin{bmatrix} R & t_c \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{2\alpha} e^{2\beta} e^{2\gamma} & t_x \\ 0 & 1 \end{bmatrix} \]

\[ = \begin{bmatrix} \cos \beta \cos \gamma - \sin \beta \sin \gamma & \sin \alpha \sin \beta \sin \gamma + \sin \alpha \cos \gamma \cos \beta & x_c \\ \sin \beta \cos \gamma + \sin \alpha \cos \beta \sin \gamma & \cos \alpha \sin \beta \sin \gamma - \cos \alpha \cos \gamma \cos \beta & y_c \\ -\sin \beta & \cos \beta & z_c \end{bmatrix}. \]  

(12)

We calculate out the coordinates of \(M_1, M_2, M_3, P\) in the moving coordinate system Cxyz: \(M_1^r(r, 0, 0)\), \(M_2^r(-1/2, \sqrt{3}/2, 0)\), \(M_3^r(-1/2, -\sqrt{3}/2, 0)\), \(P^r(0,0,n)\).

Therefore, their coordinates in the absolute coordinate system are:

\[ M_1 = RM_1^c + t_c, \]

(13)

\[ P = RP^c + t_c. \]

(14)

In solving forward kinematics in the above section, we know point \(M_1, M_2, M_3\) can be represented by \(\theta_{1i}, \theta_{2i}, \theta_{3i}\) \((i = 1 \sim 3)\). Based on the corresponding relationship, we have three square matrix equations:

\[ \begin{bmatrix} (c\theta_{1i}c\theta_{2i}c\theta_{3i} - s\theta_{1i}s\theta_{3i})m + lc\theta_{1i}c\theta_{2i} + R \\ -ms\theta_{1i}c\theta_{3i} - ls\theta_{3i} \\ -(s\theta_{1i}c\theta_{3i} + c\theta_{1i}s\theta_{3i})m - ls\theta_{1i}c\theta_{2i} \end{bmatrix} = \]

\[ \begin{bmatrix} r\alpha c\beta + x_c \\ r\alpha s\beta + y_c \\ -rs\beta + z_c \end{bmatrix}. \]  

(15)

\[ \begin{bmatrix} p_1 & q_1 & k_1 \\ p_2 & q_2 & k_2 \end{bmatrix} = \begin{bmatrix} d_1 & e_1 & f_1 \\ d_2 & e_2 & f_2 \end{bmatrix}, \]

(16)

\[ \begin{bmatrix} p_1 & q_1 & k_1 \\ p_2 & q_2 & k_2 \end{bmatrix} = \begin{bmatrix} d_1 & e_1 & f_1 \\ d_2 & e_2 & f_2 \end{bmatrix}, \]

(17)

where

\[ p_1 = \frac{\left((-c\theta_{12}c\theta_{23} + s\theta_{12}c\theta_{22} + R) + \sqrt{3} (ms\theta_{22}c\theta_{23})^{1/2} + \frac{1}{2} \right)}{2}, \]

\[ q_1 = \frac{\sqrt{3} \left((-c\theta_{12}c\theta_{23} - c\theta_{23}c\theta_{22} + R) + \frac{1}{2} \right)}{2} + \frac{1}{2}, \]

\[ k_1 = \frac{-c\theta_{12}c\theta_{22}c\theta_{23} + c\theta_{23}c\theta_{22})m - ls\theta_{21}c\theta_{22}}{2}. \]
Inverse kinematics solution of the metamorphic parallel cutting mechanism

\[ d_i = -\gamma \alpha \beta \alpha + \sqrt{\gamma} (\cos \beta \gamma \alpha - \sin \gamma \alpha) / 2 + x_C \]
\[ e_i = -\gamma \alpha \beta \alpha + \sqrt{\gamma} (\cos \beta \gamma \alpha + \sin \gamma \alpha) / 2 + y_C \]
\[ f_i = \rho \beta \alpha / 2 + \sqrt{\gamma} \cos \beta \gamma \alpha / 2 + z_C \]

So the inverse kinematics solution is completed.

TABLE 1 Inverse kinematics solution of the metamorphic parallel cutting mechanism's original configuration framework

<table>
<thead>
<tr>
<th>(C(x, y, z, \gamma, \beta, \alpha))</th>
<th>(\theta_1, \theta_2)</th>
<th>(\theta_3, \theta_4)</th>
<th>(\theta_5, \theta_6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(900,0,1800)</td>
<td>(3.445° 0°)</td>
<td>(-95.963° 23.286°)</td>
<td>(-95.963° 23.286°)</td>
</tr>
<tr>
<td>(0°,30°, 0°)</td>
<td>(-130.724° 0°)</td>
<td>(-133.34° 23.286°)</td>
<td>(-133.34° 23.286°)</td>
</tr>
<tr>
<td>(176.557° 180°)</td>
<td>(-49.377° 180°)</td>
<td>(95.963° 156.714°)</td>
<td>(95.963° -156.714°)</td>
</tr>
<tr>
<td>(11° 12°)</td>
<td>(133.34° 156.714°)</td>
<td>(133.34° -156.714°)</td>
<td>(133.34° -156.714°)</td>
</tr>
</tbody>
</table>

As the input crank rotation angles of the mechanism, we obtain eight sets of forward kinematics solutions by solving the forward kinematics equation of the mechanism. For more detailed calculation results, see Table 2. It can be seen that the seven equations of \(s_{\gamma\theta}\) values calculated out based on the seven sets of \(y\) values of equation (11) all satisfy equation (7) and thus are all reasonable solutions. Hence, we obtain eight sets of forward kinematics solutions.

TABLE 2 Forward Kinematics Solution of the Metamorphic Parallel Cutting Mechanism's Sub-configuration Framework

<table>
<thead>
<tr>
<th>No</th>
<th>(\theta_1)</th>
<th>(\theta_2)</th>
<th>(\theta_3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.5956</td>
<td>3.0572</td>
<td>3.0572</td>
</tr>
<tr>
<td>2</td>
<td>3.1422</td>
<td>3.0572</td>
<td>3.0572</td>
</tr>
<tr>
<td>3</td>
<td>1.9517+1.2135i</td>
<td>-2.2954+1.0379i</td>
<td>-2.2954+1.0379i</td>
</tr>
<tr>
<td>4</td>
<td>-2.1642+0.7489i</td>
<td>-2.2954+1.0379i</td>
<td>-2.2954+1.0379i</td>
</tr>
<tr>
<td>5</td>
<td>1.9516-0.2133i</td>
<td>-2.2954-1.0379i</td>
<td>-2.2954-1.0379i</td>
</tr>
<tr>
<td>6</td>
<td>-2.1642-0.7484i</td>
<td>-2.2954-1.0379i</td>
<td>-2.2954-1.0379i</td>
</tr>
<tr>
<td>7</td>
<td>2.5728</td>
<td>0.6784</td>
<td>0.6784</td>
</tr>
<tr>
<td>8</td>
<td>1.9045</td>
<td>0.6784</td>
<td>0.6784</td>
</tr>
</tbody>
</table>

After calculation, the obtained coordinates of \(M_1, M_2, M_3\) and \(M_4\) in the absolute coordinate system are shown in Table 3.

TABLE 3 The Coordinates of \(M_1, M_2, M_3\) and \(M_4\) in the Absolute Coordinate System

<table>
<thead>
<tr>
<th>No</th>
<th>(x_M)</th>
<th>(y_M)</th>
<th>(z_M)</th>
<th>(x_M)</th>
<th>(y_M)</th>
<th>(z_M)</th>
<th>(x_M)</th>
<th>(y_M)</th>
<th>(z_M)</th>
</tr>
</thead>
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<tr>
<td>1</td>
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<td>282.205</td>
<td>301.79</td>
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<td>282.205</td>
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<td>282.205</td>
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<td>0</td>
<td>-1418.2</td>
<td>-453.439</td>
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<td>-1418.2</td>
<td>-453.439</td>
<td>0</td>
<td>-1418.2</td>
</tr>
<tr>
<td>3</td>
<td>272.887</td>
<td>0</td>
<td>268.508</td>
<td>272.887</td>
<td>0</td>
<td>268.508</td>
<td>272.887</td>
<td>0</td>
<td>268.508</td>
</tr>
<tr>
<td>4</td>
<td>1592.8</td>
<td>0</td>
<td>1592.8</td>
<td>1592.8</td>
<td>0</td>
<td>1592.8</td>
<td>1592.8</td>
<td>0</td>
<td>1592.8</td>
</tr>
</tbody>
</table>

Based on Table 3 and the equation (15), (16) and (17), we can calculate out the coordinates of the geometric center C and the z-y-x Euler angles of the moving platform, see Table 4.
Since the phase spaces of the first and the third set of solutions will cause interference in the mechanism, and the phase space of the second set is outside the range of maximum rotation angle \( \theta_i (i = 1 \sim 3) \), we give all these three sets of solutions up. Errors resulting from using approximate solutions of inverse kinematics to solve the forward kinematics of the mechanism are small between the fourth set of solutions and the inverse kinematics solutions. Figure 5 is the abbreviated drawing of the mechanism pose of the fourth solutions set, and the corresponding model pose is shown in Figure 6.

### 3.2.1 Forward Kinematic Analysis of the Sub-configuration Framework

Figure 7 shows the moving coordinate system and kinematic screw of metamorphic parallel cutting mechanism’s the sub-configuration framework. Establish the absolute coordinate system \( OXYZ \), and the sub-configuration framework has only 3 degrees of freedom.

Suppose the angles of rotation about \( e_{1i}, e_{2i}, e_{3i}, e_{4i}, e_{5i} \) (\( i = 1 \sim 3 \)) in each limb chain are \( \theta_{1i}, \theta_{2i}, \theta_{3i}, \theta_{4i}, \theta_{5i} \) (\( i = 1 \sim 3 \)) respectively. Let metamorphic kinematic pair \( B \) be the driving joint and the rest be driven joints. Suppose the coordinates of the geometric center \( C \) of the moving platform \( M_iM_2M_3 \) in the absolute coordinate system are \( (x_C, y_C, z_C) \), based on which the coordinates of \( M_1, M_2, M_3 \) in the absolute coordinate system are:

\[
M_1 = \begin{bmatrix} x_c + r \\ y_c \\ z_c \end{bmatrix},
M_2 = \begin{bmatrix} x_c - r/2 \\ y_c + \sqrt{3}r/2 \\ z_c \end{bmatrix},
M_3 = \begin{bmatrix} x_c - r/2 \\ y_c - \sqrt{3}r/2 \\ z_c \end{bmatrix}.
\]

(18)

To calculate the coordinates of \( D_i \) in the absolute coordinate system, let fully expanded limb chain \( B_iD_iM_i \) be the initial phase space, at which the conversion between \( B_i, x_i, y_i, z_i \) and the absolute coordinate system is:

\[
g_i (\theta) = e^{\theta_i e_i} g_i (0).
\]

(19)

And the coordinates of \( D_i \) in the absolute coordinate system are:

\[
D_i = R_{1i} D_i^0 + t_{1i}.
\]

(20)

According to the rigidity constraint equation \( [D_iM_i] = m^2 (i = 1 \sim 3) \), we have the position kinematics equation of the metamorphic parallel mechanism’s sub-configuration framework:

\[
\begin{bmatrix}
(x_i + r - l \cos \theta_{1i} - R) + y_i^1 \left( z_i + l \sin \theta_{1i} \right)^2 - m^2 = 0 \\
(y_i + c)^2 + (b + d)^2 = m^2 = 0 \\
(z_i + h)^2 = m^2 = 0
\end{bmatrix}.
\]

(21)

Let \( a = r - l \cos \theta_{1i} - R \), \( b = l \sin \theta_{1i} \), \( c = l \sin \theta_{1i} \), \( h = l \sin \theta_{1i} \), \( e = r/2 + (l \cos \theta_{3i}) + R/2 \), \( d = \sqrt{3}r/2 - \sqrt{3} (l \cos \theta_{3i}) + R/2 \), \( f = -r/2 + (l \cos \theta_{3i}) + R/2 \), \( g = -\sqrt{3}r/2 + \sqrt{3} (l \cos \theta_{3i}) + R/2 \).

So that the simultaneous equations (21) can be written as:

\[
\begin{bmatrix}
(x_i + a)^2 + y_i^2 + (z_i + b)^2 - m^2 = 0 \\
(x_i + c)^2 + (y_i + d)^2 + (z_i + e)^2 - m^2 = 0 \\
(x_i + f)^2 + (y_i + g)^2 + (z_i + h)^2 - m^2 = 0
\end{bmatrix}.
\]

(22)

which can be simplified to a matrix form:

\[
\begin{bmatrix}
ad - c & -d & b & e & x_i & 0 \\
ad - f & -g & b & -h & y_i & 0 \\
a - d & c & f & g & 0 & z_i
\end{bmatrix} = \begin{bmatrix}
d^2 - a^2 + c^2 + b^2 + e^2 \\
d^2 - a^2 + f^2 + b^2 + h^2
\end{bmatrix}.
\]

(23)

Based on equation (23), let \( z_i \) represent \( x_i \) and \( y_i \), and substitute it into equation (24), thereby deriving the
analytic expression of \( z_c \) relative to \( \theta_{11}, \theta_{21} \) and \( \theta_{31} \). Then \( x_c \) and \( y_c \) can be calculated out, and the forward kinematics solution is achieved.

3.2.2 Inverse kinematic analysis of the sub-configuration framework

Substituting the half angle formula
\[
\sin \theta = 2\tan(\theta/2)/\left[1 + \tan^2(\theta/2)\right],
\]
\[
\cos \theta = [1 - \tan^2(\theta/2)]/\left[1 + \tan^2(\theta/2)\right]
\]
into equation (21), we have the expressions of \( \tan(\theta_{1i}/2), \tan(\theta_{2i}/2) \) and \( \tan(\theta_{3i}/2) \) about \( x_c, y_c, z_c \), thereby obtaining the input crank rotation angles \( \theta_{1i}, \theta_{2i} \) and \( \theta_{3i} \). Then the inverse kinematics solution of the sub-configuration framework is achieved.

3.2.3 A numerical example of forward and inverse kinematics solution

Suppose the coordinates of the cutting gun endpoint P are \((1000,0,1000)\), and the coordinates of the moving platform geometric center C are \((1000,0,800)\). Substituting these two known quantities into equation (21), we have the input crank rotation angles \( \theta_{1i}, \theta_{2i} \) and \( \theta_{3i} \). See Table 5 for detailed calculation results. The results \( \theta_{1i} = -156.033^\circ \) and \( \theta_{2i} = \theta_{3i} = -167.665^\circ \) will cause interference or unreasonable distribution of the components in the mechanism, so we give them up. Then supposing the coordinates of the moving platform geometric center C is \((-1000,0,2000)\), \( \theta_{1i} = -159.101^\circ \) and \( \theta_{2i} = \theta_{3i} = -96.178^\circ \) among the results obtained are also unreasonable and are therefore casted.

TABLE 5 Inverse kinematics solution of the metamorphic parallel mechanism’s sub-configuration framework

<table>
<thead>
<tr>
<th>( \mathbf{C}(x_c, y_c, z_c) )</th>
<th>((1000,0,800))</th>
<th>((-1000,0,200))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta_{1i} )</td>
<td>(23.174^\circ, -156.033^\circ, -159.101^\circ, -146.005^\circ)</td>
<td>(-116.654^\circ, 167.665^\circ, -96.178^\circ, 32.956^\circ)</td>
</tr>
<tr>
<td>( \theta_{2i} )</td>
<td>(-116.654^\circ, 167.665^\circ, -96.178^\circ, 32.956^\circ)</td>
<td>(-116.654^\circ, 167.665^\circ, -96.178^\circ, 32.956^\circ)</td>
</tr>
</tbody>
</table>

Taken \( (\theta_{1i}, \theta_{2i}, \theta_{3i}) = (23.174^\circ, -116.654^\circ, -116.654^\circ) \) in Table 5 as input crank rotation angles, we obtain the forward kinematics solution by solving the forward position equation of the mechanism. See Table 6 for detailed calculation results.

TABLE 6 Forward Kinematics Solution of Metamorphic Parallel Mechanism’s Sub-configuration Framework

<table>
<thead>
<tr>
<th>( \mathbf{C}(x_c, y_c, z_c) )</th>
<th>((1006.427,0.995,852))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta_{1i}, \theta_{2i}, \theta_{3i} )</td>
<td>((23.174^\circ,-116.654^\circ,-116.654^\circ))</td>
</tr>
</tbody>
</table>

Obviously, \((1006.427,0.995,852)\) are reasonable solutions; Figure 8 is the corresponding pose of end moving platform at the model mechanism; Figure 9 is the corresponding pose of the model. Taking the inverse kinematics solution results calculated to the thousandth as the input of forward kinematics solution may result in errors between the forward and the inverse kinematics solution results.

4 Study on kinematics simulation of the metamorphic parallel cutting mechanism

4.1 KINEMATICS SIMULATION OF THE METAMORPHIC PARALLEL CUTTING MECHANISM’S METAMORPHIC ORIGINAL CONFIGURATION FRAMEWORK

4.1.1 MATLAB-based kinematics numerical simulation of the original configuration framework

The sub-configuration framework of metamorphic parallel cutting mechanism should be able to perform upper and lower bevel cutting on a surface, which means the end moving platform can move with changing spatial pose and trajectory. Now we take upper bevel cutting as an example, and the space motion requirement for the cutting gun endpoint P are:

\[
\alpha = 0 \quad \beta = -\left(\frac{\pi}{6}\right) \cdot \sin\left(2\pi t / 2\right) \quad \gamma = \left(\frac{\pi}{6}\right) \cdot \cos\left(2\pi t / 2\right)
\]

where \( T \) is a time cycle with the value of 60s.

Set 300 evenly spaced numerical points \( t \) from 0 to 60s. Based on the above inverse kinematics equation of metamorphic parallel cutting mechanism’s original configuration framework, using Matlab, we calculate out the 300 numerical values of the 6 driving kinematics pairs rotation angles in the original configuration framework. Then draw the curve of driving kinematic pair rotation angles to time based on the numerical values obtained. As shown in Figure 10, compute the derivatives of the analytic expressions of the six input crank rotation angles with respect to \( t \), and obtain 300 values of inputs angular velocities. Then draw the angular velocity curve, as shown in Figure 11.
4.1.2 ADAMS-based kinematics simulation of the original configuration framework

Figure 12 shows the simulation model. Add six driving force on the three Hooke’s joints; and draw spline curve using the 300 numerical values of driving kinematic pair rotation angles calculated out by MATLAB and add it as driven spline curve to the AKISPL function. Figure 13 is the spline curve of the input crank rotation angle $\theta_1$.

Perform kinematics simulation of the above model by setting the simulation time 60s and the number of time steps 300. And measure the displacement and angle changes of the coordinate system of the end moving platform’s center of mass relative to that of the fixed platforms, thereby obtaining the angular displacement and the angular velocity change curves of the end moving platform’s center of mass, as shown in Figure 14.

4.1.3 The simulation output analysis

The curve derived from the simulation shows the ADAMS-based simulation output coincides well with the motion characteristics as well as the changing trend of the center of mass of the mechanism end moving platform with initial settings. To find the numerical errors between the simulation output and the theoretical motion characteristics, the simulation values and theoretical values of the each motion parameters at the time of $t=0, 10, 20, 30, 40, 50, 60$ are collected, thereby obtaining the maximum relative errors. See Table 7 for more detail.

<table>
<thead>
<tr>
<th>Motion Parameters Relative Errors</th>
<th>$X$(mm)</th>
<th>$Y$(mm)</th>
<th>$Z$(mm)</th>
<th>$X$-Angle(*)</th>
<th>$Y$-Angle(*)</th>
<th>$Z$-Angle(*)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Relative Errors</td>
<td>0.0048</td>
<td>0.0059</td>
<td>0.0005</td>
<td>0.006</td>
<td>0.0021</td>
<td>0.0044</td>
</tr>
</tbody>
</table>
Comparing the ADAMS simulation values and theoretical values of motion parameters of the metamorphic parallel cutting mechanism’s original configuration framework, as shown in Table 7, we know the maximum relative errors at each time point are all restricted on a level of 0.01. It means the kinematics method adopted in this paper for the study of the metamorphic parallel cutting mechanism’s original configuration framework is correct and reasonable, and the bevel cutting on the surface can be achieved via this mechanism.

4.2 KINEMATICS SIMULATION OF THE METAMORPHIC PARALLEL CUTTING MECHANISM’S SUB-CONFIGURATION FRAMEWORK

4.2.1 MATLAB-based kinematics numerical simulation of the sub-configuration framework

The metamorphic parallel cutting mechanism’s sub-configuration framework has 3 degrees of freedom, and the motion of the end moving platform is 3D satisfying the requirement of square groove cutting on a surface.

Now we give the specific requirements for trajectory of the cutting gun endpoint:

\[
\begin{align*}
x &= 1000\sin \left( \frac{2\pi t}{T} \right) \\
y &= 1000\cos \left( \frac{2\pi t}{T} \right) \\
z &= -400\sin \left( \frac{2\pi t}{T} \right) + 1400
\end{align*}
\]

where \( T \) is a time cycle with value of 60s.

Set 300 evenly spaced numerical points \( t \) from 0 to 60s. Based on the inverse kinematics equation of the sub-configuration framework, using Matlab, we calculate out the input crank rotation angles and angular velocities at the 300 numerical points, and then draw the curve of input rotation angle and angular velocity to time, as shown in Figure 15 and Figure 16.

4.2.2 ADAMS-based kinematics simulation of the sub-configuration framework

The kinematics simulation method applied to the study of the metamorphic parallel cutting mechanism’s sub-configuration framework is the same with that applied to the original configuration. And Figure 17 shows the simulation model. Using the same method in section 3.1.2, import the values of the three driving pair rotation angles obtained from MATLAB into ADAMS, and use Spline curve to fit the data points of driving pair rotation angles. Then, set spline-driven pattern for the sub-configuration framework.

Perform kinematics simulation of the above model by setting the simulation time 60s and the number of time steps 300. And measure the displacement and angle changes of the coordinate system of the center of mass of the end moving platform relative to that of the fixed platforms, thereby obtaining the angular displacement and the angular velocity change curves of the center of mass of the end moving platform, as shown in Figure 18.

4.2.3 The simulation output analysis

The analysing method is the same as that applied to the original configuration framework. Similarly, the ADAMS simulation values and theoretical values (motion values of the end moving platform’s center of mass in the mechanism with initial settings) of the each motion parameters at the time of \( t=0,10,20,30,40,50,60 \)s are collected, thereby obtaining the maximum relative errors as shown in Table 8.

<table>
<thead>
<tr>
<th>Motion Parameters</th>
<th>X(mm)</th>
<th>Y(mm)</th>
<th>Z(mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Relative Errors</td>
<td>0.0028</td>
<td>0.0045</td>
<td>0.0013</td>
</tr>
</tbody>
</table>

Comparing the ADAMS simulation value and theoretical value (the mechanism with initial settings) of the motion parameters of the metamorphic parallel cutting mechanism’s sub-configuration framework, as
shown in Table 8, we know the maximum relative errors at each time point are all restricted to a level of 0.01. It means the kinematics method adopted in this paper for the study of sub-configuration framework is correct and reasonable, and the bevel cutting on the surface can be achieved via this mechanism.

5 Conclusions

(1) Based on the traditional spherical pair, a novel metamorphic kinematic pair with four different frameworks was designed according to the principles of metamorphic mechanism. And then we devised a metamorphic parallel cutting mechanism structure using this kinematic pair.

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