Equilibrium distributions of the queue length in M/M/c queuing system

Jian Xu*

School of Economics and Management, Beijing University of Posts and Telecommunications, Beijing, 100876, China

Received 1 July 2014, www.tsi.lv

Abstract

In this paper, An M/M/c queuing system with multiple working vacations and vacation interruption is considered. All servers work at a lower rate rather than completely stop during a vacation period. Meanwhile, we introduce another vacation policy: vacation interruption. Otherwise using matrix-geometric solution method, we obtain steady-state distribution for queue length.

Keywords: M/M/c; working vacation; matrix-geometric solution

1 Introduction

1.1 LITERATURE REVIEW

During the last two decades, the queuing systems with server vacations or working vacations have been investigated extensively due to their applications in various fields, such as computer systems, communication networks, production managing, etc. General vacation models can be found in Tian and Zhang [1].

About the study of working vacations, in 2002, Servi and Finn [2] first studied an M/M/1 queue with working vacations (Such model is denoted by M/M/1/WV queue), where inter-arrival times, service times during service period, service times during vacation period, and vacation times are all exponentially distributed. They developed the explicit formulae for the mean and variance number of customers in the system. Later in [3], Wu and Takagi extended Servi and Finn's M/M/1/WV queue to an M/G/1/WV queue. They assumed that service times during service period, service times during vacation period as well as vacation times are all generally distributed. Further, they assumed that when a working vacation ends, if there are customers in the system, the server changes to another service rate, where the service times follow a different distribution. In [4], Baba extended Servi and Finn's M/M/1/WV queue to a GI/M/1/WV queue. They not only assumed general independent arrival, they also assumed service times during service period, service times during vacation period as well as vacation times following exponential distribution. Furthermore, Baba derived the steady- state system length distributions at arrival and arbitrary epochs. In [5], Banik et al. studied a finite capacity GI/M/1 queue with multiple working vacations and presented a series of numerical results. For more comprehensive and excellent study on the working vacation models, the readers may refer to [6-8] and references therein for details.

For the Multi-server vacation models, there are only a limited number of studies due to the complexity of the systems. The M/M/c queue with exponential vacations was first studied by Levy and Yechiali [9]. Chao and Zhao [10] investigated a GI/M/c vacation system and provided an algorithm to compute the performance measures. Tian et al. [11] gave a detailed study of the M/M/c vacation systems in which all servers take multiple vacation policy when the system is empty. Later, Zhang and Xu [12], Zhang and Tian [13] and Ke et al. [14] analysed the M/M/c vacation systems with a "partial server multiple vacation policy".

Existing research works about multi-server vacation models, including those mentioned above, have not related to "the working vacation policy". Besides the lack of research work on this problem, the existing literature about "vacation Interruption" also focuses on a single server model. Li and Tian [15] first introduced and studied an M/M/1 queue. Using the matrix analytic method, Li and Tian [16] generalized their results to the discrete-time GI/Geo/1 queue. The continuous-time GI/M/1queue was analysed by Li et al. [17]. Using the method of a supplementary variable, Zhangand Hou [18] investigated the M/G/1 queue with working vacations and vacation interruption.

1.2 CHARACTERISTICS OF OUR MODEL

The model we consider has some certain implications in practice. In some situations, the number of servers is not one so that a more general model should be used. Therefore, compared with previous studies, our model is more general, and an M/M/1 queue system can be seen as a special case of M/M/c queue system, that is $c=1$. In the cyclic service queue system, which is always used to

* ***Corresponding author*** e-mail: xunjian311987@126.com

reconfigure the communication network, we can adopt the working vacation policy to model. Moreover, the analysis of our model can also provide the theory and analysis method to design the optimal lower service rate.

The rest of this paper is organized as follows. In Section 2, we give a brief description of the mathematical model, and a quasi-birth and death (QBD) process is demonstrated. In Section 3, we compute the stationary distributions by using the matrix geometric solution method, and the conditional stochastic decomposition properties are proved in this part. Finally, Section4 summarizes the investigation and draws the conclusion.

2 Mathematical model and QBD process

We consider the M/M/c/ queue with Working Vacations and vacation Interruption, and the specific application about the "Working Vacations and vacation Interruption"policy in this paper is introduced in 1.2 (Characteristics of our model). It is assumed that customers arrive according to a Poisson process with rate λ.The service times during busy period follow exponential distribution with mean $1/\mu_1$. The service times during vacation period follow another exponential distribution with mean $1/\mu_2$. Vacation times are exponentially distributed with mean $1/\theta$. The service order is assumed to be First Come First Served. In addition, inter-arrival times, service times, and vacation times are mutually independent.

Let $Q_v(t)$ be the number of customers in the system at time t, let $J(t)$ be the indicator variable defined by:

$$J(t)=\begin{cases}0, & \text{all servers are on a busy period at time } t\\ 1, & \text{all setvers are on a working period at time } t\end{cases}.$$

Then, the $\{Q_v(t), J(t)\}$ is a quasi-birth–death process (QBD) with the state space:

$$\Omega=\{0,1\}\cup\{(k,j),k\ge 1, j=0,1\}. \quad (1)$$

Referring to the state-transition-rate diagram as shown in Figure 1 the infinitesimal generator Q of the QBD describing the M/M/c queuing system with Working Vacations and vacation Interruption is of the form:

$$Q=\begin{array}{c} k=0\\ k=1\\ k=2\\ \cdots\\ \cdots\\ k=c-1\\ k=c\\ k=c+1\\ k=c+2\\ \cdots \end{array}\begin{bmatrix} A_0 & C_0 & & & & & & & \\ B_1 & A_1 & C_1 & & & & & & \\ & B_2 & A_2 & C_2 & & & & & \\ & & \cdots & \cdots & \cdots & & & & \\ & & & \cdots & \cdots & \cdots & & & \\ & & & & B_{c-1} & A_{c-1} & C_{c-1} & & \\ & & & & & B_c & A_c & C_c & \\ & & & & & & B_{c+1} & A_{c+1} & C_{c+1} \\ & & & & & & & B_{c+2} & A_{c+2} & C_{c+2} \\ & & & & & & & & \cdots & \cdots & \cdots \end{bmatrix} \quad (2)$$

The entries $B_k(0\le k\le c)$, $A_k(0\le k\le c)$ and $C_k(0\le k\le c)$ are different matrices, define:

$A_0=\lambda, C_0=(0,\lambda), B_1=(\mu_1,\mu_2)^T$.

Other matrices are 2×2 square matrices, and $C=C_k=\lambda I(1\le k\le c)$, where I is an identity matrix

$$A_k=\begin{bmatrix}-(\lambda+k\mu_1) & \theta\\ \theta & -(\lambda+\theta+k\mu_2)\end{bmatrix},(1\le k\le c-1),$$

$$A=A_k=\begin{bmatrix}-(\lambda+c\mu_1) & \theta\\ \theta & -(\lambda+\theta+c\mu_2)\end{bmatrix},(k\ge c),$$

$$B_k=\begin{bmatrix}k\mu_1 & 0\\ k\mu_2 & 0\end{bmatrix},(2\le k\le c-1),$$

$$B=B_k=\begin{bmatrix}c\mu_1 & 0\\ c\mu_2 & 0\end{bmatrix},(k\ge c).$$

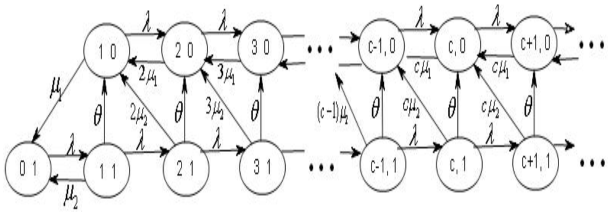

FIGURE 1 State-transition-rate diagram

Note that Q is also viewed as the infinitesimal generator for the QBD process. To analyse this QBD process, a very import ant matrix in evaluating the performance measures is the matrix R. It is known as the rate matrix, and it is the minimal non-negative solution of the matrix quadratic equation (the readers are referred to Neuts [19]):

$$R^2B+RA+C=0. \quad (3)$$

Based on the structures of matrices, A and C, which are represented as the lower triangular matrix, thus the matrix solution R is also the lower triangular matrix.

Doing some arduous algebraic derivations and arrangement, we develop the explicit formula for matrix R in the theorem below. Firstly, let's assume that $\rho=\dfrac{\lambda}{c\mu_1}$.

Theorem 1. If $\rho<1$, the matrix Equation (3) has the minimal nonnegative solution as $R=\begin{bmatrix}\rho & 0\\ r_{21} & r_{22}\end{bmatrix}$, where:

$$r_{21}=\frac{\lambda(\lambda+\theta)}{c\mu_1(c\mu_2+\lambda+\theta)}, r_{22}=\frac{\lambda}{c\mu_2+\lambda+\theta}.$$

Proof: Based on the structures of matrices A, B and C we can assume that R has the same structure as:

$$R=\begin{bmatrix}\rho & 0\\ r_{21} & r_{22}\end{bmatrix}.$$

Substituting R into the matrix Equation (3), we obtain the following system of equations:

$$\begin{cases} c\mu_1 r_{11}^2 - r_{11}(c\mu_1+\lambda)+\lambda=0 & (4.1)\\ -r_{22}(c\mu_2+\lambda+\theta)+\lambda=0 & (4.2)\\ c\mu_2 r_{22}^2 + c\mu_1 r_{21}^2(r_{11}+r_{22}) - r_{21}(c\mu_1+\lambda)=0 & (4.3)\end{cases}. \quad (4)$$

To obtain the minimal nonnegative solution of Equation (3), in the first equation of the system of Equations (4.1), let $r_{11}=\rho$ (the other root is $r_{11}=1$). In the second equation of the system of equations (4.2), we can get $r_{22}=\dfrac{\lambda}{c\mu_2+\lambda+\theta}$.

Substituting $r_{11}=\rho$ and $r_{22}=\dfrac{\lambda}{c\mu_2+\lambda+\theta}$ into the third Equation (4.3), we get:

$$r_{21}=\frac{c\mu_2 r_{22}^2+r_{22}\theta}{c\mu_1(1-r_{22})}=\frac{\lambda(\lambda+\theta)}{c\mu_1(c\mu_2+\lambda+\theta)} \text{ and } 0<r_{21}<1.$$

In fact: $r_{21}=\dfrac{\lambda(\lambda+\theta)}{c\mu_1(c\mu_2+\lambda+\theta)}=\dfrac{\lambda}{c\mu_1}\dfrac{\lambda+\theta}{c\mu_2+\lambda+\theta}$.

When $\rho=\dfrac{\lambda}{c\mu_1}<1$, we can get $r_{21}<1$.

Furthermore, we can verify that all diagonal elements of rate matrix R are less than 1. Therefore, the spectral radius of rate matrix R, $SP(R)=\max\{\rho,r_{22}\}$ is less than 1. Based on the ttheorem (Neuts [19]), we can prove that the QBD process $\{Q_v(t),J(t)\}$ is positive recurrent if and only if $\rho<1$.

3 Stationary distributions

$\rho<1$ is the stability condition of the state process. Using the rate matrix R, we can solve the steady-state probability more efficiently. Let (Q_v,J) be a set of random variables which follows the stationary distribution of the QBD process $\{Q_v(t),J(t)\}$. Define:

$\Pi=(\pi_0,\pi_1,\pi_2,\ldots)$ and $\pi_0=(\pi_{01}),\pi_k=(\pi_{k0},\pi_{k1}),k\ge 1$.

$$\pi_{kj}=P\{Q_v=k,J=j\}=$$
$$\lim_{t\to\infty}P\{Q_v(t)=k,J(t)=j\},(k,j)\in\Omega$$

So the stationary distribution for this QBD process is given as follows:

Theorem2. If $\rho<1$, the distribution of (Q_v,J) is:

$$\pi_{m,1}=$$
$$\begin{cases}\displaystyle\prod_{i=1}^{m}\frac{\lambda}{i\mu_2+\lambda+\theta}K, \quad 1\le m\le c-1 & (5.1)\\ \displaystyle\prod_{i=1}^{c-1}\frac{\lambda}{i\mu_2+\lambda+\theta}\left(\frac{\lambda}{c\mu_2+\lambda+\theta}\right)^{m-c+1}K, \quad m\ge c & (5.2)\end{cases}, (5)$$

$$\pi_{m,0}=\begin{cases}\displaystyle K\frac{1}{m!}\left(\frac{\lambda}{\mu_1}\right)^m\varphi_m, \quad 1\le m\le c-1 & (6.1)\\ \displaystyle\pi_{c-1,0}\rho^{m-1+1}+\pi_{c-1,1}r_{21}\sum_{i=0}^{m-c}r_{11}^i r_{22}^{m-c-i}, \quad m\ge c & (6.2)\end{cases}. \quad (6)$$

$$K=(K_1+K_2+K_3)^{-1},$$

where:

$$\varphi_m=1-\frac{\mu_2}{\mu_2+\lambda+\theta}-\alpha-\beta+\sum_{p=1}^{c-1}p!\left(\frac{\mu_1}{\lambda}\right)^p,$$

$$\alpha=\sum_{k=2}^{m}\sum_{j=1}^{k}\frac{j(k-1)!\mu_1^{k-1}\mu_2}{\lambda^{k-j}\prod_{i=1}^{j}(i\mu_2+\lambda+\theta)},$$

$$\alpha=\sum_{k=2}^{m}\sum_{j=1}^{k}\frac{\theta(k-1)!\mu_1^{k-1}}{\lambda^{k-j}\prod_{i=1}^{j}(i\mu_2+\lambda+\theta)},$$

$$r_{11}=\frac{\lambda}{c\mu_1},r_{22}=\frac{\lambda}{c\mu_2+\lambda+\theta},r_{21}=\frac{\lambda(\lambda+\theta)}{c\mu_1(c\mu_2+\lambda+\theta)},$$

$$K_1=\sum_{m=1}^{c-1}\left[\prod_{i=1}^{m}\frac{\lambda}{i\mu_2+\lambda+\theta}+\frac{1}{m!}\frac{\lambda}{\mu_1}\varphi_m\right],$$

$$K_2=\sum_{m=c}^{\infty}\left[\prod_{i=1}^{c-1}\frac{\lambda}{i\mu_2+\lambda+\theta}+\left(\frac{\lambda}{c\mu_2+\lambda+\theta}\right)^{m-c+1}\right],$$

$$K_3=\sum_{m=c}^{\infty}\left[\frac{1}{(c-1)!}\left(\frac{\lambda}{\mu_1}\right)^{c-1}\varphi_{c-1}\rho^{m-c+1}+\right.$$
$$\left.\prod_{i=1}^{c-1}\frac{\lambda}{i\mu_2+\lambda+\theta}r_{21}\sum_{i=1}^{m-c}r_{11}^i r_{22}^{m-c-i}\right]$$

Proof: Using the matrix geometric solution method(see[19]),we can get:

$$\pi_m=\pi_{c-1}R^{m-c+1},m\ge c,$$

the equations that $\pi_{01}, \pi_{10}, \pi_{11}, \pi_{20}, \pi_{21}, ..., \pi_{c-1,0}, \pi_{c-1,1}$ satisfy are:

$$\begin{cases} -\lambda\pi_{01} + \mu_1\pi_{10} + \mu_2\pi_{11} = 0 & (7.1) \\ -(\lambda+\mu_1)\pi_{10} + \theta\pi_{11} + 2\mu_1\pi_{20} + 2\mu_2\pi_{20} = 0 & (7.2) \\ \lambda\pi_{01} - (\lambda+\theta+\mu_2)\pi_{11} & (7.3) \\ \lambda\pi_{10} - (\lambda+2\mu_1)\pi_{20} + \theta\pi_{21} + 3\mu_1\pi_{30} + 3\mu_2\pi_{31} & (7.4) \\ \lambda\pi_{11} - (\lambda+\theta+2\mu_2)\pi_{21} & (7.5) \\ \lambda\pi_{20} - (\lambda+3\mu_1)\pi_{20} + \theta\pi_{31} + 4\mu_1\pi_{40} + 4\mu_2\pi_{41} & (7.6) \\ \lambda\pi_{21} - (\lambda+\theta+3\mu_2)\pi_{31} & (7.7) \\ \dots & \\ \lambda\pi_{c-3,0} - [\lambda+(c-2)\mu_1]\pi_{c-2,0} + \theta\pi_{c-2,1} + & \\ (c-1)\mu_1\pi_{c-1,0} + (c-1)\mu_2\pi_{c-1,1} = 0 & (7.8) \\ \lambda\pi_{c-3,1} - [\lambda+\theta+(c-2)\mu_2]\pi_{c-2,1} = 0 & (7.9) \\ \lambda\pi_{c-2,1} - [\lambda+\theta+(c-1)\mu_2]\pi_{c-1,1} = 0 & (7.10) \end{cases} \quad (7)$$

Define: $\pi_{01} = K$.

Through a series of calculation, we can get the following result from the Equation (7) {(7.3),(7.5),(7.7),…,(7.10)}.

$$\pi_{11} = \frac{\lambda}{\lambda+\theta+\mu_2}K, \quad (7.3a)$$

$$\pi_{21} = \frac{\lambda}{\lambda+\theta+2\mu_2}\pi_{11} = \frac{\lambda}{\lambda+\theta+2\mu_2}\frac{\lambda}{\lambda+\theta+\mu_2}K, \quad (7.5a)$$

$$\pi_{31} = \frac{\lambda}{\lambda+\theta+3\mu_2}\pi_{21} = \frac{\lambda}{\lambda+\theta+3\mu_2}\frac{\lambda}{\lambda+\theta+2\mu_2}\frac{\lambda}{\lambda+\theta+\mu_2}K, \quad (7.7a)$$

…

$$\pi_{c-1,1} = \frac{\lambda}{\lambda+\theta+(c-1)\mu_2}\pi_{c-2,1} = \prod_{i=1}^{c-1}\frac{\lambda}{\lambda+\theta+i\mu_2}K. \quad (7.10a)$$

Clearing up the above formulas, we can obtain Equation (5.1).

Meanwhile, we can get the following result from Equations (7) {(7.2), (7.4), (7.6),…, (7.8)}

$$2\mu_1\pi_{20} - \lambda\pi_{10} = \mu_1\pi_{10} - \theta\pi_{11} - 2\mu_2\pi_{21} \quad (7.2a)$$

$$3\mu_1\pi_{30} - \lambda\pi_{20} = 2\mu_1\pi_{20} - \lambda\pi_{10} - \theta\pi_{21} - 3\mu_2\pi_{31} \quad (7.4a)$$

$$4\mu_1\pi_{40} - \lambda\pi_{30} = 3\mu_1\pi_{30} - \lambda\pi_{20} - \theta\pi_{31} - 4\mu_2\pi_{41} \quad (7.6a)$$

…

$$(c-1)\mu_1\pi_{c-1,0} - \lambda\pi_{c-2,0} = (c-2)\mu_1\pi_{c-2,0} - \lambda\pi_{c-3,0} - \theta\pi_{c-2,1} - (c-1)\mu_2\pi_{c-1,1} \quad (7.8a)$$

The above formulas on both sides of the equal are added, respectively. Through a series of calculation, we can get:

$$(c-1)\mu_1\pi_{c-1,0} - \lambda\pi_{c-2,0} = \mu_1\pi_{10} - \theta\pi_{11} - 2\mu_2\pi_{21} - \theta\pi_{21} - ... - \theta\pi_{c-2,1} - (c-1)\mu_2\pi_{c-1,1} \quad (8)$$

By (7.1), we can get:

$$\mu_1\pi_{10} = \lambda\pi_{01} - \mu_2\pi_{11}. \quad (7.1a)$$

Substituting Equation (7.1a) into Equation (8), we obtain Equation (6.1).

And $R = \begin{bmatrix} \rho & 0 \\ r_{21} & r_{22} \end{bmatrix}$, so we can calculate:

$$R^{m-1+1} = \begin{bmatrix} \rho^{m-c+1} & 0 \\ r_{21}\sum_{i=0}^{m-c} r_{11}^i r_{22}^{m-c-i} & r_{22}^{m-c+1} \end{bmatrix}.$$

Substituting $\pi_{c-1,0}$ (5.1), $\pi_{c-1,1}$ (6.1) and R^{m-c+1} into $(\pi_{m,0}, \pi_{m,1}) = (\pi_{c=1,0}, \pi_{c-1,1})R^{m-c+1}$, we can obtain Equations (5.2) and (6.2).

Finally by the normalization conditions can be obtained, we can get π_{01}, i.e K.

4 Conclusions

In this paper, we study an M/M/c queuing system with multiple working vacation and vacation interruption using matrix-analytic method. This system is formulated as a QBD process, the necessary and sufficient condition for the stability of the system was deduced. More important, the explicit solution of stable condition and rate matrix of the QBD model are obtained, and then the stationary probability distributions are explicitly developed.

References

[1] Tian N, Zhang Z 2006 Vacation Queueing Models – Theory and Applications *Springer-Verlag* New York

[2] Servi L, Finn S 2002 M/M/1 queue with working vacations (M/M/1/WV) *Perform Eval* **50** 41–52

[3] Wu D, Takagi H 2006 M/G/1 queue with multiple working vacations *Perform Eval* **63** 654–81

[4] Baba Y 2005 Analysis of a GI/M/1 queue with multiple working vacations *Oper Res Lett* **33** 201–9

[5] Banik A D, Gupta U C, Pathak S S 2007 On the GI/M/1/N queue with multiple vacation-analytic analysis and computation *Appl Math Model* **31** 1701–10

[6] Jain M, Sharma G C, Sharma R 2011 Working vacation queue with service interruption and multi optional repair *Int J Inform Manage Sci* **22** 157–75
[7] Ke J C, Wu C H, Zhang Z G 2010 Recent developments in vacation queueing models: a short survey *Int J Oper Res* **7** 3–8
[8] Zhang M, Hou Z 2010 Performance analysis of M/G/1 queue with working vacations and vacation interruption *J Comput Appl Math* **234** 2977–85
[9] Levy Y, Yechiali U 1976 An M/M/c queue with servers' vacations, INFOR **14** 153–63
[10] Chao X, Zhao Y 1998 Analysis of multiple-server queues with station and server vacations *European Journal of Operational Research* **110** 392–406
[11] Tian N, Xu X 2001 M/M/c queue with synchronous multiple vacation of partial servers *OR Trans* **5** (3) 85–94
[12] Xu X, Zhang Z G 2006 Analysis of multiple-server queue with a single vacation (e, d)–policy *Performance Evaluation* **63** 825–38
[13] Zhang Z G , Tian N 2003 Analysis on queueing systems with synchronous vacations of partial servers *Performance Evaluation* **52** 269–82
[14] Ke J-C, Lin C H, Yang J Y, Zhang Z G 2009 Optimal (d, c) vacation policy for a finite buffer M/M/c queue with unreliable servers and repairs *Applied Mathematical Modeling* **33** 3949–62
[15] Li J, Tian N 2007 The M/M/1 queue with working vacations and vacation interruption *J Syst Sci Syst Eng* **16** 121–27
[16] Li J, Tian N 2007 The discrete-time GI/Geo/1 queue with working vacations and vacation interruption *Appl Math Comput* **185** 1–10
[17] Li J, Tian N, Ma Z 2008 Performance analysis of GI/M/1 queue with working vacations and vacation interruption *Appl Math Modell* **32** 2715–30
[18] Zhang M, Hou Z 2010 Performance analysis of M/G/1 queue with working vacations and vacation interruption *J Comput Appl Math* **234** 2977–85
[19] Neurs M 1981 Matrix-Geometric Solution in Stochastic Models *Johns Hopkins University Press* Baltimore 1981
[20] Fuhrmann S W, Cooper R B 1985 Stochastic decomposition in the M/G/1 queue with generalized vacations *Oper Res* **33** 1117–29
[21] Miyazawa N 1994 Decomposition formulas for single server queues with vacations: An unified approach *Stochastic Models* **10** 389–413

Author

Jian Xu, born in 1987, China

University studies: PhD in School of Economics and Management at Beijing University of Posts and Telecommunications in China.
Scientific interest: supply chain management, operations research and optimization, queuing theory, logistics management.