An M/G/1 retrial queue subject to disasters and N-policy vacation

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Abstract

An M/G/1 retrial queue subject to disasters and N-policy vacation is investigated in this paper. Both positive and negative customers arrival in Poisson processes independently, and positive customers receive service immediately if the server is idle upon their arrivals. Otherwise, they enter a retrial orbit and repeat their attempt again after a random time period. Once negative customers arrive, they not only remove all the customers in the system but also make the server under repair. The server leaves for an N-policy vacation as soon as the system empties. By using the supplementary variables method, we obtain the steady-state solutions for both performance measures and reliability quantities.

Keywords: Retrial queues, Disasters, N-policy Vacations, Reliability

1 Introduction

Queueing models with negative arrivals were first introduced by Gelenbe(1989)[1], which are also called G-queues. Negative arrivals have been interpreted as viruses, orders of demand, or inhibitors, which affect the queue behavior in a variety of ways. For example, an arriving negative customer removes all the customers in the system. We regard this kind of negative customer as a disaster (catastrophe). Disasters are also connected with other queueing phenomena such as clearing systems [2]. There has been an increasing interest in queueing systems and networks with disasters due to their applications in telecommunication, computer networks and manufa-cturing systems. Numerous papers have recently appeared in which a disaster removes all the customers in the system [3-6]. Artalejo & Gomez-Corral (1999) [4] computation of the limiting distribution in queueing systems with repeated attempts and disasters, Li and Lin (2006) [5] analyzed an M/G/1 processor-sharing queue with disasters by means of extending the supplementary variable method. Wang, Liu and Li (2008) [6] further analyzed an M/G/1 retrial queue subject to disasters and server failures. Gao and Wang (2014) [7] studied the performance and reliability of an M/G/1-G retrial queue with orbital search.

Several authors have analyzed the N-policy on queueing systems with server vacation. Kella (1989) [8] provided detailed discussions concerning N-policy. Choudhury & Paul (2004) [9] studied a batch arrival queue with an additional service channel under N-policy. Liu, Wu and Yang (2009) [10] analyzed an M/G/1 retrial G-queue under N-policy where the arrival of a negative customer only removes the customer being in service from the system. Wu, Tang and Yu (2014) [11] further analyzed an M/G/1 queue with multiple vacations, N-policy and unreliable service.

In this paper, we mainly consider an M/G/1 retrial queue under N-policy vacation subject to disasters, server failures and repairs.Our model is related to computer networks operating under the presence of viruses, whereas disaster has the effect of clearing operation that kills all messages present in the network and also destroys the network itself. Positive arrivals receive service immediately if the server is idle upon their arrivals. Otherwise, they may enter a retrial orbit and try their luck after a random time interval. Negative customers affect the system when the system is on service or on retrial, and they not only remove all the customers in the system, but also make the server breakdowns. If the server is on repair or on vacation, negative customers have no effect to the system. In the N-policy for the retrial queue, an idle server is activated the moment N customers accumulated in the orbit.

The rest of this paper is arranged as follows. In the next section, we give the mathematical description of the considered retrial queueing models. The stationary differential equations of the model and their solutions are obtained in section 3. The reliability quantities are analyzed in section 4. Finally, a conclusion is presented in section 5.

2 Mathematical model descriptions

In this paper, we consider an M/G/1 stochastic clearing system with repeated attempts, where the server applies N-policy vacation when no customer is recorded in the system.

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2.1 THE ARRIVAL PROCESS

We assume that the system has two types of independent arrivals, positive and negative customers. Both positive and negative customers arrive at a single server system according to two independent Poisson processes with rates λ^+ and λ^-, respectively.

2.2. THE RETRIAL RULE AND SERVICE

If an arriving positive customer finds the server idle, the customer gets service immediately. Otherwise, the server is found busy or broken or on N-policy vacation, the positive customer will join a group of unsatisfied customers (i.e. orbit) and repeats its request for service after some random time until he finds the server idle. Successive inter-retrial times of customers are independently, exponentially distributed with parameter δ. The service time distribution function is given by

$$B(x)=1-\exp\{-\int_0^x \mu(t)dt\} \text{ with } b_1\in(0,+\infty).$$

2.3. THE REMOVAL RULE AND REPAIR

When the server is idle or busy, the arrival of a negative customer not only removes all the customers of the system (including the customer being in service and the customers in the orbit), but also makes the server fails immediately. The server is sent for repair immediately once it fails, and after repair it is as good as new. As soon as the repair of the server is complete, the server is idle if there is one more customer in the retrial orbit. Otherwise, the server enters the N-policy vacation. An arriving negative customer has no effect to the system when the server is on vacation or on repair. We call arrived negatives which actually affect the system as disasters in this paper. The repair time distribution function is given by

$$R(x)=1-\exp\{-\int_0^x \varphi(t)dt\} \text{ with mean } r_1\in(0,+\infty).$$

2.4 THE N-POLICY VACATION

When the server finishes serving a positive customer or completing its repair, the server finds the system empty, the server enters the N-policy vacation (the dormant period) until N positive customers accumulated. If the server finds no customers in the orbit, it enters the idle period.

2.5 THE INDEPENDENCE

We assume that all the random variables defined above are independent and we denote $\overline{F}(x)=1-F(x)$ as the tail of distribution $F(x)$. We also denote:

$$\lambda=\lambda^+ +\lambda^-, F^*(x)=\int_0^{+\infty} e^{-sx}dF(x),$$

$$\overline{F}(x)=\int_0^{+\infty} e^{-sx}\overline{F}(x)dx=\frac{1-F^*(s)}{s}.$$

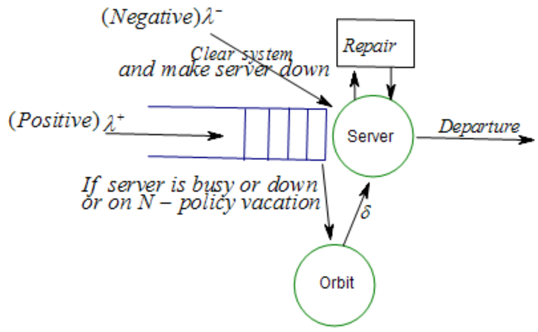

Figure 1 The structure of the system

3 The differential equations and the solution

In this section, we first introduce several supplementary variables to construct the differential equations for our model. Then we solve the equations and derive the probability generating functions (PGFs) of the stationary orbit size distribution and some interesting performance measures of the system.

We define the states of the server as

$$C(t)=\begin{cases}0, \text{ if the server is in a dormant period at time t,}\\ 1, \text{ if the server is busy at time t,}\\ 2, \text{ if the server is idle at time t,}\\ 3, \text{ if the server is under repair at time t.}\end{cases}$$

For $t\ge 0$, we define $N(t)$ as the number of customers in the orbit at time t and we further define the random variable $\xi(t)$ as follows: If $C(t)=1$, $\xi(t)$ represents the elapsed service time up to time *t*; if $C(t)=3$, $\xi(t)$ denotes the elapsed repair time up to time *t*. Then $\{C(t),N(t),\xi(t),t\ge 0\}$ becomes a Markov chain.

We can define:

$$V_n(t)=P\{C(t)=0,N(t)=n\},t\ge 0,0\le n<N,$$
$$S_n(x,t)dx=P\{C(t)=1,N(t)=n,x<\xi(t)<x+dx\},\quad t\ge 0,x>0,n\ge 0,$$

$$I_n(t)=P\{C(t)=2,N(t)=n\},t\ge 0,n\ge 1,$$
$$R_n(x,t)dx=P\{C(t)=3,N(t)=n,x<\xi(t)<x+dx\},\quad t\ge 0,x>0,n\ge 0.$$

If the system is stable, then we further have:

$$V_n=\lim_{t\to\infty}V_n(t),\quad S_n(x)=\lim_{t\to\infty}S_n(x,t),\quad I_n=\lim_{t\to\infty}I_n(t),$$

$$R_n(x)=\lim_{t\to\infty}R_n(x,t).$$

3.1 THE STEADY STATE EQUATIONS

If the system is in stable state, the system of stationary differential equations of the joint probability density $\{V_n,S_n(x),R_n(x),n\ge 0;I_n,n\ge 1\}$ can be written as:

$$\lambda^+V_0=\int_0^\infty S_0(x)\mu(x)dx+\int_0^\infty R_0(x)\varphi(x)dx, \tag{1}$$

$$\lambda^+ V_n = \lambda^+ V_{n-1}, \quad 1 \le n < N, \tag{2}$$

$$\frac{dS_n(x)}{dx} = -(\lambda + \mu(x))S_n(x) + \lambda^+(1-\delta_{n,0})S_{n-1}(x), n \ge 0, \tag{3}$$

$$(\lambda + n\delta)I_n = \int_0^\infty S_n(x)\mu(x)dx + \int_0^\infty R_n(x)\varphi(x)dx + \lambda^+\delta_{n,N}V_{n-1}, \quad n \ge 1 \tag{4}$$

$$\frac{dR_n(x)}{dx} = -(\lambda^+ + \varphi(x))R_n(x) + \lambda^+(1-\delta_{n,0})R_{n-1}(x), n \ge 0 \tag{5}$$

The boundary conditions are given below:

$$S_n(0) = \lambda^+(1-\delta_{n0})I_n + (n+1)\delta I_{n+1}, \quad n \ge 0, \tag{6}$$

$$R_0(0) = \lambda^-[\int_0^\infty S_n(x)dx + (1-\delta_{n0})I_n], \tag{7}$$

$$R_n(0) = 0, \quad n \ge 1, \tag{8}$$

where $\delta_{i,j}$ denotes Kronecker's delta function. The normalization condition is

$$\sum_{n=1}^{\infty} I_n(x) + \sum_{n=0}^{N-1} V_n(x) + \sum_{n=0}^{\infty}\int_0^\infty S_n(x)dx + \sum_{n=0}^{\infty}\int_0^\infty R_n(x)dx = 1. \tag{9}$$

3.2 THE MODEL SOLUTION

To solve the system displayed in Eq. (1) – (9), we can define the following PGFs for $|z| \le 1$.

$$I(z) = \sum_{n=1}^{\infty} I_n z^n, \qquad V(z) = \sum_{n=0}^{N-1} V_n z^n,$$

$$S(x,z) = \sum_{n=0}^{\infty} S_n(x) z^n, \qquad R(x,z) = \sum_{n=0}^{\infty} R_n(x) z^n.$$

It follows from (2) that:

$$V(z) = V_0 \sum_{n=0}^{N-1} z^n = \frac{V_0(1-z^N)}{1-z}. \tag{10}$$

It follows from (3) and (5) that:

$$S(x,z) = S(0,z)\exp\{(\lambda^+ z - \lambda)x\}\overline{B}(x), \tag{11}$$

$$R(x,z) = R(0,z)\exp\{(\lambda^+ z - \lambda^+)x\}\overline{R}(x), \tag{12}$$

where $S(0,z)$ and $R(0,z)$ can be obtained from (6)-(8).

Multiplying Eq. (6)-(8) by z^n and then summing over all possible values of $n \in z^+$, we get:

$$S(0,z) = \lambda^+ I(z) + \delta I'(z), \tag{13}$$

$$R(0,z) = \lambda^-[\int_0^\infty S(x,z)dx + I(z)] = R_0(0) = R(0,0).$$

Putting (11) and (13) into above equation, and simplifying it, we get:

$$R(0,z) = \lambda^-[S(0,z)\overline{B}(\lambda - \lambda^+ z) + I(z)] = R(0,0)$$
$$= \lambda^- S(0,0)\overline{B}(\lambda) = \lambda^-\delta I'(0)\overline{B}(\lambda). \tag{14}$$

Now substituting (13) and (14) into (11) and (12) respectively, we have:

$$S(x,z) = [\lambda^+ I(z) + \delta I'(z)]\exp\{(\lambda^+ z - \lambda)x\}\overline{B}(x), \tag{15}$$

$$R(x,z) = \lambda^-\delta I'(0)\overline{B}(\lambda)\exp\{(\lambda^+ z - \lambda^+)x\}\overline{R}(x), \tag{16}$$

It follows from (5) and (14) that:

$$R_0(x) = R_0(0)\exp\{-\lambda^+ x\}\overline{R}(x) = R(0,0)\exp\{-\lambda^+ x\}\overline{R}(x)$$
$$= \lambda^-\delta I'(0)\overline{B}(\lambda)\exp\{-\lambda^+ x\}\overline{R}(x),$$

which leads to:

$$\int_0^\infty R_0(x)\varphi(x)dx = \lambda^-\delta I'(0)\overline{B}(\lambda)R^*(\lambda^+). \tag{17}$$

It follows from (3) and (13) that:

$$S_0(x) = S_0(0)\exp\{-\lambda x\}\overline{B}(x) = S(0,0)\exp\{-\lambda x\}\overline{B}(x)$$
$$= \delta I'(0)\exp\{-\lambda x\}\overline{B}(x),$$

which leads to:

$$\int_0^\infty S_0(x)\mu(x)dx = \delta I'(0)B^*(\lambda). \tag{18}$$

Substituting (17) and (18) into (1), we get:

$$V_0 = \frac{\delta I'(0)}{\lambda^+}[B^*(\lambda^+) + \lambda^-\overline{B}(\lambda)R^*(\lambda^+)]. \tag{19}$$

Substituting (19) into (10), we have:

$$V(z) = \frac{\delta I'(0)[B^*(\lambda^+) + \lambda^-\overline{B}(\lambda)R^*(\lambda^+)](1-z^N)}{\lambda^+(1-z)}$$
$$= I'(0)M(z), \tag{20}$$

where $M(z) = \frac{\delta[B^*(\lambda^+) + \lambda^-\overline{B}(\lambda)R^*(\lambda^+)](1-z^N)}{\lambda^+(1-z)}$.

Now we try to find the expression of $I(z)$.

From (1),(2) and (4), we have:

$$\lambda^+(1-z)V(z) + \lambda I(z) + \delta z I'(z)$$
$$= \int_0^\infty S(x,z)\mu(x)dx + \int_0^\infty R(x,z)\varphi(x)dx. \tag{21}$$

Substituting (15), (16) and (20) into (21), we get:

$$I'(z) = I(z)D(z) + E(z),$$

where

$$E(z) = I'(0)H(z),$$

$$D(z) = -\frac{\lambda}{\delta[z - B^*(\lambda - \lambda^+ z)]},$$

$$H(z)=-\frac{(1-z^N)}{z-B^*(\lambda-\lambda^+z)}\{B^*(\lambda)+\lambda^-\overline{B}(\lambda)R^*(\lambda^+)\}$$

$$+\frac{\lambda^-I'(0)\overline{B}(\lambda)R^*(\lambda^+-\lambda^+z)}{z-B^*(\lambda-\lambda^+z)}$$

Then we have:

$$I(z)=\exp\{\int_0^z D(t)dt\}\int_0^z E(u)\exp\{-\int_0^u D(t)dt\}du$$
$$=I'(0)\exp\{\int_0^z D(t)dt\}\int_0^z H(u)\exp\{-\int_0^u D(t)dt\}du$$
$$=I'(0)L(z),$$

where $L(z)=\exp\{\int_0^z D(t)dt\}\int_0^z H(u)\exp\{-\int_0^u D(t)dt\}du$.

Now we need to find $I'(0)$. Summarizing above equations, we have:
$V(z)=I'(0)M(z)$,
$E(z)=I'(0)H(z)$,
$I(z)=I'(0)L(z)$,
then we further have:

$$V(1)=I'(0)M(1),\qquad E(1)=I'(0)H(1),$$

$$I(1)=I'(0)L(1),\qquad I'(1)=I'(0)[D(1)L(1)+H(1)],$$

where

$$M(1)=\frac{N\delta[B^*(\lambda^+)+\lambda^-\overline{B}(\lambda)R^*(\lambda^+)]}{\lambda^+},$$

$$H(1)=\frac{\lambda^-I'(0)\overline{B}(\lambda)}{1-B^*(\lambda^-)},$$

$$D(1)=-\lambda/\{\delta[1-B^*(\lambda^-)]\},$$

$$L(1)=\exp\{\int_0^1 D(t)dt\}\int_0^1 H(u)\exp\{-\int_0^u D(t)dt\}du.$$

It follows from (15) and (16) that:

$$\int_0^\infty S(x,1)dx=[\lambda^+I(1)+\delta I'(1)]\overline{B}(\lambda^-),$$

$$\int_0^\infty R(x,1)dx=\lambda^-\delta I'(0)]\overline{B}(\lambda^-)r_1.$$

From the normalization condition (9), we obtain:

$$I(1)+V(1)+\int_0^\infty S(x,1)dx+\int_0^\infty R(x,1)dx=1.$$

Finally, substituting above expressions into the normalization condition, we get:

$I'(0)=\dfrac{1}{W}$, where

$$W=L(1)+M(1)+\{\lambda^+L(1)+\delta[D(1)L(1)+H(1)]\}\overline{B}(\lambda^-)$$
$$+\lambda^-r_1\overline{B}(\lambda).$$

Thus we summarize above main results in the following Theorem 3.1.

Theorem 3.1 If the system is stable, then the joint distribution of the number in the orbit and the server's state has the following PGFs

$$I(z)=\frac{L(z)}{W},\ V(z)=\frac{M(z)}{W},$$

$$S(z)=\frac{\{\lambda^+L(z)+\delta[D(z)L(z)+H(z)]\}\overline{B}(\lambda-\lambda^+z)}{W},$$

$$R(z)=\frac{\lambda^-\delta\overline{B}(\lambda)\overline{R}(\lambda^+-\lambda^+z)}{W},$$

where

$$L(z)=\exp\{\int_0^z D(t)dt\}\int_0^z H(u)\exp\{-\int_0^u D(t)dt\}du,$$

$$M(z)=\frac{\delta[B^*(\lambda^+)+\lambda^-\overline{B}(\lambda)R^*(\lambda^+)](1-z^N)}{\lambda^+(1-z)},$$

$$D(z)=-\frac{\lambda}{\delta[z-B^*(\lambda-\lambda^+z)]},$$

$$H(z)=-\frac{(1-z^N)}{z-B^*(\lambda-\lambda^+z)}\{B^*(\lambda)+\lambda^-\overline{B}(\lambda)R^*(\lambda^+)\}$$

$$+\frac{\lambda^-\overline{B}(\lambda)R^*(\lambda^+-\lambda^+z)}{W[z-B^*(\lambda-\lambda^+z)]},$$

$$W=L(1)+M(1)+\{\lambda^+L(1)+\delta[D(1)L(1)+H(1)]\}\overline{B}(\lambda^-)$$
$$+\lambda^-r_1\overline{B}(\lambda)$$

Next we are interested in studying the marginal orbit size distributions when the system in steady-state.

Theorem 3.2 If the system is stable, the marginal PGFs of the server's state orbit size distribution are given by

$$S(z)=\int_0^{+\infty}S(x,z)dx=\frac{\{\lambda^+L(z)+d[D(z)L(z)]\}\overset{=}{B}(\lambda-\lambda^+z)}{W},$$

$$R(z)=\int_0^{+\infty}R(x,z)dx=\frac{\lambda^-\delta\overset{=}{B}(\lambda)\overset{=}{R}(\lambda^+-\lambda^+z)}{W},$$

and the expressions of $I(z)$ and $V(z)$ are given in Theorem 3.1.

Note that $I(z)$ is the probability generating function of orbit size when the server is idle, $V(z)$ is the probability generating function of the orbit size when the server is on vacation, $S(z)$ is the probability generating function of the orbit size when the server is on service, $R(z)$ is the probability generating function of the orbit size when the server is on repair.

The system in steady state probabilities and some performance measures are given in the following corollary.

Corollary 3.1 If the system is stable, then:
(1) The probability that the server is idle is:

$$P_I=\lim_{z\to1}I(z)=\frac{1}{W}\exp\{\int_0^1 D(t)dt\}\int_0^1 H(u)\exp\{-\int_0^u D(t)dt\}du.$$

(2) The probability that the server is in dormant period is:

$$P_V = \lim_{z\to 1} V(z) = \frac{N\delta[B^*(\lambda^+) + \lambda^- \overline{B}(\lambda) R^*(\lambda^+)]}{\lambda^+ W}.$$

(3) The probability that the server is in service is:

$$P_s = \lim_{z\to 1} S(z) = \frac{\{\lambda^+ L(1) + \delta[D(1)L(1) + H(1)]\}\overline{B}(\lambda^-)}{W}.$$

(4) The probability that the server is under repair is:

$$P_R = \lim_{z\to 1} R(z) = \frac{\lambda^- \delta \overline{B}(\lambda) r_1}{W}.$$

(5) The steady state failure frequency of the server is given by:

$$W_f = \lambda^-[I(1) + \int_0^{+\infty} S(x,1)\,\mathrm{d}\,x]$$
$$= \frac{\lambda^-[L(1) + \{\lambda^+ L(1) + \delta[D(1)L(1) + H(1)]\}\overline{B}(\lambda^-)]}{W}.$$

(6) The probability generating function of the orbit size distribution at the stationary state is given by:

$$P(z) = V(z) + I(z) + S(z) + R(z)$$
$$= \frac{1}{W}\{L(z) + \{\lambda^+ L(z) + \delta[D(z)L(z) + H(z)]\}\overline{B}(\lambda - \lambda^+ z)$$
$$+ \frac{\delta[B^*(\lambda^+) + \lambda^- \overline{B}(\lambda) R^*(\lambda^+)](1 - z^N)}{\lambda^+(1-z)}$$
$$+ \lambda^- \delta \overline{B}(\lambda) \overline{R}(\lambda^+ - \lambda^+ z)\}.$$

(7) The probability generating function of the system size distribution at the stationary state is given by:

$$Q(z) = V(z) + I(z) + zS(z) + R(z)$$
$$= \frac{1}{W}\{L(z) + z\{\lambda^+ L(z) + \delta[D(z)L(z) + H(z)]\}\overline{B}(\lambda - \lambda^+ z)$$
$$+ \frac{\delta[B^*(\lambda^+) + \lambda^- \overline{B}(\lambda) R^*(\lambda^+)](1 - z^N)}{\lambda^+(1-z)}$$
$$+ \lambda^- \delta \overline{B}(\lambda) \overline{R}(\lambda^+ - \lambda^+ z)\}.$$

(8) The probability generating function of the orbit size distribution at the stationary point of time is given by:

$$J = V'(1) + I'(1) + S'(1) + R'(1).$$

(9) The expected number of customers in the system is:

$$K = J + S(1).$$

Theorem 3.3 If the system is stable, the PGF of the steady state distribution of the number of customers in the system at a departure epoch of this model is given by:

$$\Pi(z) = \frac{\{\lambda^+ L(z) + \delta[D(z)L(z) + H(z)]\}\overline{B}(\lambda - \lambda^+ z)}{\{\lambda^+ L(1) + \delta[D(1)L(1) + H(1)]\}\overline{B}(\lambda^-)}.$$

Proof. Following the argument of PASTA, we note that a departing customer in the system just after a departure if and only if there were j customers in the system before the departure. We denote $\{\pi_j, j \in z^+\}$ as the probability that there are j customers in the system at a departure epoch. Then for $j \in z^+$ we have:

$$\pi_j = K_0 \int_0^\infty S_j(x)\mu(x)dx,$$

where K_0 is the normalizing constant.

Then multiplying both sizes of above equation by z^j and taking summation over $j \in z^+$, we have:

$$\Pi(z) = K_0 \int_0^\infty S(x,z)\mu(x)dx$$
$$= \frac{K_0\{\lambda^+ L(z) + \delta[D(z)L(z) + H(z)]\}\overline{B}(\lambda - \lambda^+ z)}{W}. \qquad (22)$$

Utilizing normalizing condition $\Pi(1) = 1$, we get:

$$K_0 = \frac{W}{\{\lambda^+ L(z) + \delta[D(z)L(z) + H(z)]\}\overline{B}(\lambda - \lambda^+ z)},$$

and substituting K_0 into (22), we get Theorem 3.3 immediately.

Theorem 3.4 Let B be the length of a generalized busy period (excluding the dormant period). Then, under the steady-state, we have

$$E(B) = \frac{\lambda^+ W - \delta N[B^*(\lambda^+) + \lambda^- \overline{B}(\lambda) R^*(\lambda^+)]}{\lambda^+ \delta[B^*(\lambda^+) + \lambda^- \overline{B}(\lambda) R^*(\lambda^+)]}.$$

Proof. By applying the argument of an alternating renewal process, we have:

$$P_V = \frac{E(V)}{E(V) + E(B)}, \quad E(V) = \frac{N}{\lambda^+},$$

then we can easily get $E(B) = \frac{N(1 - P_V)}{\lambda^+ P_V}$, i.e.,

$$E(B) = \frac{\lambda^+ W - \delta N[B^*(\lambda^+) + \lambda^- \overline{B}(\lambda) R^*(\lambda^+)]}{\lambda^+ \delta[B^*(\lambda^+) + \lambda^- \overline{B}(\lambda) R^*(\lambda^+)]}.$$

4 Reliability of the server

Denote the time to the first failure of the server by τ, then the reliability function of the server is $\zeta(t) = P(\tau > t)$. In order to get the reliability of the server, letting the failure states of the server be absorbing states, then we obtain a new retrial queueing system. In the new system, we use the same notations as in section 3, and then we can get the following differential equations:

$$\frac{dV_0(t)}{dt} = -\lambda^+ V_0(t) + \int_0^\infty S_0(x)\mu(x)dx, \qquad (23)$$

$$\frac{dV_n(t)}{dt} = -\lambda^+ V_n(t) + \lambda^+ V_{n-1}(t), \quad 1 \le n < N, \tag{24}$$

$$\frac{\partial S_n(x,t)}{\partial x} + \frac{\partial S_n(x,t)}{\partial t} = -(\lambda + \mu(x)) S_n(x,t) + \lambda^+ (1-\delta_{n,0}) S_{n-1}(x,t), n \ge 0, \tag{25}$$

$$\frac{dI_n(t)}{dt} = -(\lambda + n\delta) I_n(t) + \int_0^\infty S_n(x,t)\mu(x)dx + \lambda^+ \delta_{n,N} V_{n-1}(t), \ n \ge 1, \tag{26}$$

and the boundary conditions:

$$S_n(0,t) = \lambda^+ (1-\delta_{n,0}) I_n(t) + (n+1)\delta I_{n+1}(t), \ n \ge 0, \tag{27}$$

and the initial conditions:

$$V_n(0) = \delta_{0,n}, S_n(x,0) = 0, I_n(0) = 0, \tag{28}$$

by taking Laplace transforms of above equations, we obtain:

$$s\tilde{V}_0(s) - 1 = -\lambda^+ \tilde{V}_0(s) + \int_0^\infty \tilde{S}_0(x,s)\mu(x)dx, \tag{29}$$

$$s\tilde{V}_n(s) = -\lambda^+ \tilde{V}_n(s) + \lambda^+ \tilde{V}_{n-1}(s),, \quad 1 \le n < N, \tag{30}$$

$$s\tilde{S}_n(x,t) + \frac{\partial \tilde{S}_n(x,s)}{\partial x} = -(\lambda + \mu(x)) \tilde{S}_n(x) + \lambda^+ (1-\delta_{n,0}) \tilde{S}_{n-1}(x,s), n \ge 0, \tag{31}$$

$$s\tilde{I}_n(s) = -(\lambda + n\delta) \tilde{I}_n(s) + \int_0^\infty \tilde{S}_n(x,s)\mu(x)dx + \lambda^+ \delta_{n,N} \tilde{V}_{n-1}(s), \ n \ge 1, \tag{32}$$

$$\tilde{S}_n(0,s) = \lambda^+ (1-\delta_{n,0}) \tilde{I}_n(s) + (n+1)\delta \tilde{I}_{n+1}(s), \ n \ge 0. \tag{33}$$

We define the following generating functions:

$$\tilde{I}(z,s) = \sum_{n=1}^{\infty} \tilde{I}_n(s) z^n, \quad \tilde{V}(z,s) = \sum_{n=0}^{N-1} \tilde{V}_n(s) z^n,$$

$$\tilde{S}(x,z,s) = \sum_{n=0}^{\infty} \tilde{S}_n(x,s) z^n.$$

Then it follows from (30) that:

$$\tilde{V}_n(s) = \tilde{V}_{n-1}(s)\frac{\lambda^+ z}{s+\lambda^+} = \tilde{V}_0(s)\left(\frac{\lambda^+ z}{s+\lambda^+}\right)^n, \ 1 \le n < N,$$

hence, we have:

$$\tilde{V}(z,s) = \sum_{n=0}^{N-1} \tilde{V}_0(s)\left(\frac{\lambda^+ z}{s+\lambda^+}\right)^n = \frac{[(s+\lambda^+)^N - (\lambda^+ z)^N]\,\tilde{V}_0(s)}{(s+\lambda^+)^{N-1}(s+\lambda^+ - \lambda^+ z)}. \tag{34}$$

It follows from (31), (33) and (35) respectively that:

$$\tilde{S}(z,x,s) = \tilde{S}(z,0,s)\exp\{-(s+\lambda-\lambda^+ z)x\}\overline{B}(x), \tag{35}$$

$$\tilde{S}(z,0,s) = \lambda^+ \tilde{I}(z,s) + \delta\frac{\partial \tilde{I}(z,s)}{\partial z}, \tag{36}$$

$$\tilde{S}(z,x,s) = [\lambda^+ \tilde{I}(z,s) + \delta\frac{\partial \tilde{I}(z,s)}{\partial z}] \cdot \exp\{-(s+\lambda-\lambda^+ z)x\}\overline{B}(x), \tag{37}$$

It follows from (29) and (37) that:

$$\begin{aligned} s\tilde{V}_0(s) - 1 &= -\lambda^+ \tilde{V}_0(s) + \int_0^\infty \tilde{S}_0(x,s)\mu(x)dx \\ &= -\lambda^+ \tilde{V}_0(s) + \int_0^\infty \tilde{S}(0,x,s)\mu(x)dx \\ &= -\lambda^+ \tilde{V}_0(s) + \delta \tilde{I}'(0,s) B^*(s+\lambda), \end{aligned}$$

where $\tilde{I}'(0,s) = \frac{\partial \tilde{I}(0,s)}{\partial z}\big|_{z=0}$.

Hence, we have:

$$\tilde{V}_0(s) = \frac{1 + \delta \tilde{I}'(0,s) B^*(s+\lambda)}{s+\lambda^+}.$$

From (34), we have:

$$\tilde{V}(z,s) = \frac{1 + \delta \tilde{I}'(0,s) B^*(s+\lambda)}{s+\lambda^+} \cdot \frac{(s+\lambda^+)^N - (\lambda^+ z)^N}{(s+\lambda^+)^{N-1}(s+\lambda^+ - \lambda^+ z)}. \tag{38}$$

It follows from (29), (30), (32), (37), (38) that:

$$\frac{\partial \tilde{I}(z,s)}{\partial z} = \tilde{I}(z,s)\tilde{D}(z,s) + \tilde{E}(z,s), \tag{39}$$

where

$$\tilde{D}(z,s) = \frac{s+\lambda-\lambda^+ B^*(s+\lambda-\lambda^+ z)}{\delta[B^*(s+\lambda-\lambda^+ z)-1]},$$

$$\tilde{E}(z,s) = \frac{(s+\lambda-\lambda^+ z)\tilde{V}(z,s) - 1}{\delta[B^*(s+\lambda-\lambda^+ z)-1]}.$$

Solving the first-order differential of Eq. (39), we get:

$$\tilde{I}(z,s) = exp\{\int_0^z \tilde{D}(t,s)\,\mathrm{d}t\}\int_0^z \tilde{E}(u,s)\,\mathrm{d}t\} \cdot exp\{-\int_0^u D(t,s)\,\mathrm{d}t\}du \tag{40}$$

Hence, we can obtain the following theorem.

Theorem 4.1 The Laplace transforms of $\zeta(t)$ is given by:

$$\tilde{\zeta}(s)=\frac{[1+\delta\tilde{I}'(0,s)B^*(s+\lambda)]}{(s+\lambda^+)^N}+exp\{\int_0^1\tilde{D}(t,s)\,dt\}\int_0^1\tilde{E}(u,s)\,dt\}exp\{-\int_0^u D(t,s)\,dt\}du+[\lambda^+\tilde{I}(1,s)+\delta\tilde{I}'(0,s)]\tilde{\tilde{B}}(s+\lambda^-).$$

Proof. From equations (37), (38) and (40), we obtain:

$$\tilde{S}(1,s)=[\lambda^+\tilde{I}(1,s)+\delta I'(1,s)]\tilde{\tilde{B}}(s+\lambda^-),\ \text{where}\ \tilde{I}'(1,s)=\frac{\partial\tilde{I}(z,s)}{\partial z}\Big|_{z=z},$$

$$\tilde{V}(1,s)=\frac{[1+\delta\tilde{I}'(0,s)B^*(s+\lambda)][(s+\lambda^+)^N-(\lambda^+)^N]}{s(s+\lambda^+)^N},\ \tilde{I}(1,s)=exp\{\int_0^1\tilde{D}(t,s)\,dt\}\int_0^1\tilde{E}(u,s)\,dt\}exp\{-\int_0^u D(t,s)\,dt\}du.$$

By direct calculating the expression $\tilde{\zeta}(s)=\tilde{S}(1,s)+\tilde{V}(1,s)+\tilde{I}(1,s)$,we can obtain the above Theorem 4.1.

Corollary 4.1 The mean time to the first failure (MTTFF) of the server is given by:

$$MTTFF=\int_0^{+\infty}\varsigma(t)dt=\tilde{\varsigma}(s)|_{s=0}=$$

$$=\frac{N[1+\delta\tilde{I}'(0,0)B^*(\lambda)]}{\lambda^+}+[\lambda^+\tilde{I}(1,0)+\delta I'(1,0)]\tilde{\tilde{B}}(\lambda^-)+exp\{\int_0^1\tilde{D}(t,0)dt\}\int_0^1\tilde{E}(u,0)dt\}exp\{-\int_0^u D(t,0)dt\}du.$$

5 Conclusion

In the foregoing analysis, an M/G/1 retrial queue with disasters and N-policy vacation is considered, where disasters not only clear all the system customers but also make the server broken. When the server fails, it is repaired immediately. By using the supplementary variables method, we find the generating functions of the number of customers in the orbit and in the system as well as some performance measures of the system under steady-state. In addition, we obtain the explicit expre-ssions of the reliability function of the server.

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