Finite-time synchronization of unified chaotic system

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Abstract

This paper presents finite-time synchronization for the unified chaotic system. The master-slave system synchronization is achieved within a pre-specified convergence time. Based on the Lyapunov stability theory and the finite-time stability theory, the finite-time controller is derived to make the state of two unified chaotic systems synchronized within finite-time. At the same time, the state of the slave system exponentially synchronizes state of the master system. At last numerical simulations are presented to shows the effectiveness of theoretical analysis.

Keywords: finite-time synchronization, Lyapunov stability theory, unified chaotic system, finite-time controller

1 Introduction

In 1963 Lorenz found the first attractor and chaos theory was developed [1]. Then many new chaotic systems and methods are successively found. Especially the nonlinear dynamics of chaotic system as well as chaos control and chaos synchronization have been researched by many researchers, as in [2-6].

The main idea of synchronization is to make the state of the slave system follow the state of the master system asymptotically. The master system and slave system may have identical or completely different structures. From a practical point of view, it will be more reasonable to realize synchronization in a given time. To achieve faster convergent in control systems, finite-time control is a very useful technique. Moreover, the finite-time control technique has demonstrated better robustness and disturbance rejection properties. Finite-time stability is of finite-time control. The solutions of an asymptotic system reach the equilibrium point. To illustrate this fact, let us consider the following scalar system [7].

$$\dot{x} = -x^{\frac{1}{3}}, x(0) = x_0.$$

Which solution of trajectory is unique in forward time and described by

$$x(t) = \begin{cases} (x_0^{\frac{2}{3}} - \frac{2}{3}t)^{\frac{3}{2}}, 0 \le t \le \frac{3}{2}x_0^{\frac{2}{3}} \\ 0, t \ge \frac{3}{2}x_0^{\frac{2}{3}} \end{cases}$$

The solution converges to the equilibrium $x = 0$ within a finite time. This example suggests that in order to achieve finite-time stability, non-smooth or at least non-Lipchitz continuous feedback must be employed, even if the controlled plant $\dot{x} = f(x,u,t)$ is smooth.

Many research works have been done on chaos synchronization base on finite time [8-11]. Basically, two haotic systems are synchronized by controller, which is equivalent to the asymptotic stability of the error system. Many methods have been developed to stabilize the chaotic system on finite time [12-19]. In this paper, finite-time synchronization is investigated for the unified chaotic system. Based on finite-time stability theory and Lyapunov stability theory, the controller is derived to make the state of two the unified chaotic system synchronizes within finite-time. At the same time, the state of the response system exponentially synchronizes state of the drive system with exponential rate.

2 Problem description

Considering the following master-slave system which contains the unified chaotic system.

Master system

$$\begin{aligned} \dot{x}_1 &= (25\theta + 10)(x_2 - x_1), \\ \dot{x}_2 &= (28 - 35\theta)x_1 + (29\theta - 1)x_2 - x_1 x_3, \\ \dot{x}_3 &= x_1 x_2 - \frac{(8+\theta)x_3}{3}. \end{aligned} \tag{1}$$

Slave system

$$\begin{aligned} \dot{y}_1 &= (25\theta + 10)(y_2 - y_1) + u_1, \\ \dot{y}_2 &= (28 - 35\theta)y_1 + (29\theta - 1)y_2 - y_1 y_3 + u_2, \\ \dot{y}_3 &= y_1 y_2 - \frac{(8+\theta)y_3}{3} + u_3. \end{aligned} \tag{2}$$

where $\theta \in [0,1]$. System (1) is chaotic for $\theta \in [0,1]$. When $\theta \in [0,0.8]$, system (1) reduces to the general Lorenz system; when $\theta = 0.8$, it becomes the general Lu system; and $\theta \in (0.8,1]$, system (1) is the general Chen system [7].

In this paper, we design controller $u(t) = [u_1(t), u_2(t), u_3(t)]^T$ such that the states $y_1(t), y_2(t)$ and

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$y_3(t)$ of the slave system (2) track the states $x_1(t), x_2(t)$ and $x_3(t)$ of the master system (1), respectively. The state of the slave system (2) track state of the master system (1) within a guaranteed convergence time *T*. In other words, the controller $u(t)$ is designed to achieve exponential synchronization and the finite convergence time is calculated too.

3 Preliminary definition and lemma

Let the synchronous error vector between the master system (1) and slave system (2) be defined as

$$e(t) = \left[e_1(t)e_2(t)e_3(t)\right]^T =$$
$$\left[\mathrm{y}_1(t) - x_1(t), \mathrm{y}_2(t) - x_2(t), \mathrm{y}_3(t) - x_3(t)\right]^T \tag{3}$$

Before deriving the main theorem, there is some precise definition to be given as follows.

Define 1: The slave system (2) exponentially synchronizes the master system (1), if there exists a tracking control $u(t,R)$ and positive constant h and α such that the synchronous error satisfies

$$\|e(t)\| \le h \times \exp(-\alpha t), \forall t \ge 0$$

Here, the constant α is called the exponential convergence rate.

Define 2: Consider the master system (1) and slave system (2) to realize finite-time synchronization. Then if there exists a positive value T such that

$$\lim_{x \to T} \|e(t)\| = 0 \tag{4}$$

and $\|e(t)\| = 0$, if $t \ge T$, then master-slave synchronization of system (1) and system (2) is achieved within the finite-time T.

Lemma 1: Suppose that a continuous and positive definite function $W(t)$ is a system's Lyapunov function candidate. If $W(t)$ satisfies the following differential inequality,

$$\dot{W}(t) \le -\alpha W(t) - \beta W^{\omega}(\mathrm{t}), \forall \mathrm{t} \ge 0, \mathrm{W}(0) > 0, (5)$$

and $W(t) = 0, \forall t \ge T$, then the finite time T is given by

$$\mathrm{T} = \left[\ln((\alpha \mathrm{W}^{1-\omega}(0)/\beta) + 1)/\alpha(1-\omega)\right] \tag{6}$$

Then the system is exponential finite-time stable, where $\alpha, \beta > 0$ and $0 < \omega < 1$ are constants. In the Equation (6) α and β are called the exponential and finite-time convergence rates, respectively.

Lemma 2: The following inequality holds

$$-(\mathrm{c}_1\mathrm{a}_1^2 + \mathrm{c}_2\mathrm{a}_2^2 + \cdots + \mathrm{c}_n\mathrm{a}_n^2) \le -\mathrm{c}_h(\mathrm{a}_1^2 + \mathrm{a}_2^2 + \cdots + \mathrm{a}_n^2), \tag{7}$$

where $c_i (i = 1, 2, \ldots, n)$ are positive numbers and $c_h = \min\{c_1, c_2, \ldots, c_n\}$.

Lemma 3: The following inequality holds

$$\begin{aligned} &-(\mathrm{d}_1\mathrm{a}_1^\mathrm{b} + \mathrm{d}_2\mathrm{a}_2^\mathrm{b} + \cdots + \mathrm{d}_n\mathrm{a}_n^\mathrm{b}) \le -\mathrm{d}_h(\mathrm{a}_1^\mathrm{b} + \mathrm{a}_2^\mathrm{b} + \cdots + \mathrm{a}_n^\mathrm{b}) \\ &\le -\mathrm{d}_h(\mathrm{a}_1 + \mathrm{a}_2 + \cdots + \mathrm{a}_h)^\mathrm{b} \end{aligned} \tag{8}$$

where $d_i (i = 1, 2, \ldots, n)$ are positive numbers, $a_i (i = 1, 2, \ldots, n)$ are positive numbers, $d_h = \min\{d_1, d_2, \ldots, d_n\}$, $0 < b < 1$.

Proof: When $0 < b < 1$, $a_i (i = 1, 2, \ldots, n)$ are positive numbers and $d_h = \min\{d_1, d_2, \ldots, d_n\}$, we can obtain

$$\mathrm{d}_1\mathrm{a}_1^\mathrm{b} + \mathrm{d}_2\mathrm{a}_2^\mathrm{b} + \cdots + \mathrm{d}_n\mathrm{a}_n^\mathrm{b} \ge \mathrm{d}_h\mathrm{a}_1^\mathrm{b} + \mathrm{d}_h\mathrm{a}_2^\mathrm{b} + \cdots + \mathrm{d}_h\mathrm{a}_n^\mathrm{b}$$
$$= \mathrm{d}_h(\mathrm{a}_1^\mathrm{b} + \mathrm{a}_2^\mathrm{b} + \cdots + \mathrm{a}_n^\mathrm{b}).$$

Based on the Lemma 2 of paper [8], we can deduce that

$$\mathrm{d}_h(\mathrm{a}_1^\mathrm{b} + \mathrm{a}_2^\mathrm{b} + \cdots + \mathrm{a}_n^\mathrm{b}) \ge \mathrm{d}_h(\mathrm{a}_1 + \mathrm{a}_2 + \cdots + \mathrm{a}_n),$$

so

$$-(\mathrm{d}_1\mathrm{a}_1^\mathrm{b} + \mathrm{d}_2\mathrm{a}_2^\mathrm{b} + \cdots + \mathrm{d}_n\mathrm{a}_n^\mathrm{b}) \le -\mathrm{d}_h(\mathrm{a}_1^\mathrm{b} + \mathrm{a}_2^\mathrm{b} + \cdots + \mathrm{a}_n^\mathrm{b}) \le$$
$$-\mathrm{d}_h(\mathrm{a}_1 + \mathrm{a}_2 + \cdots + \mathrm{a}_h)^\mathrm{b}.$$

Lemma 1: The controller $u(t)$ of the slave system (2) satisfies

$$\begin{aligned} &u_1 = -(25\theta + 10)(y_2 - y_1) + (25\theta + 10)(x_2 - x_1) - k_1e_1 - \beta_1 e_1^{\frac{q}{p}}, \\ &u_2 = -(28 - 35\theta)y_1 - (29\theta - 1)y_2 + y_1y_3 + (28 - 35\theta)x_1 + \\ &(29\theta - 1)x - \mathrm{x}_1x_3 - k_2e_2 - \beta_2 e_2^{\frac{q}{p}}, \\ &u_3 = -y_1y_2 + \frac{8+\theta}{3}y_3 + x_1x_2 - \frac{8+\theta}{3}x_3 - k_3e_3 - \beta_3 e_3^{\frac{q}{p}}, \end{aligned} \tag{9}$$

where $k_i (i = 1, 2, 3) > 0$, $\beta_i (i = 1, 2, 3) > 0$, p and q are two positive odd integers and $p > q$.

Proof: From Equations (1)-(3), we deduce that, for every $t \ge 0$,

$$\begin{aligned} &\dot{e}_1(t) = -k_1e_1(t) - \beta_1 e_1^{\frac{q}{p}}, \\ &\dot{e}_2(t) = -k_2e_2(t) - \beta_2 e_2^{\frac{q}{p}}, \\ &\dot{e}_3(t) = -k_3e_3(t) - \beta_3 e_3^{\frac{q}{p}}. \end{aligned} \tag{10}$$

Choose the following Lyapunov function

$$W(t) = \frac{1}{2}(e_1^2(t) + e_2^2(t) + e_3^2(t)). \tag{11}$$

The time derivative of $W(t)$ along the trajectories of the closed-loop system (11)

$$\dot{W}(t) = e_1(t)\dot{e}_1(t) + e_2(t)\dot{e}_2(t) + e_3(t)\dot{e}_3(t). \tag{12}$$

Substituting Equation (10) into Equation (12)

$$
\begin{aligned}
&\dot{W}(t)=e_1(t)\left[-k_1e_1(t)-\beta_1e_1^{\frac{q}{p}}\right]+e_2(t)\left[-k_2e_2(t)-\beta_2e_2^{\frac{q}{p}}\right]+e_3(t)\left[-k_3e_3(t)-\beta_3e_3^{\frac{q}{p}}\right]=\\
&-k_1e_1^2(t)-k_2e_2^2(t)-k_3e_3^2(t)-\beta_1e_1^{\frac{p+q}{p}}(t)-\beta_2e_2^{\frac{p+q}{p}}(t)-\beta_3e_3^{\frac{p+q}{p}}(t)\le\\
&-k_h(e_1^2(t)+e_2^2(t)+e_3^2(t))-\beta_h(e_1^{\frac{p+q}{p}}(t)+e_2^{\frac{p+q}{p}}(t)+e_3^{\frac{p+q}{p}}(t))=\\
&-k_h(e_1^2(t)+e_2^2(t)+e_3^2(t))-\beta_h((e_1^2(t))^{\frac{p+q}{2p}}+(e_2^2(t))^{\frac{p+q}{2p}}+(e_3^2(t))^{\frac{p+q}{2p}})\le\\
&-2k_h\left[\frac{1}{2}(e_1^2(t)+e_2^2(t)+e_3^2(t))\right]-2^{\frac{p+1}{2p}}\beta_h\frac{1}{2^{\frac{p+1}{2p}}}(e_1^2(t)+e_2^2(t)+e_3^2(t))^{\frac{p+q}{2p}}=\\
&-2k_h\left[\frac{1}{2}(e_1^2(t)+e_2^2(t)+e_3^2(t))\right]-2^{\frac{p+1}{2p}}\beta_h\left[\frac{1}{2}(e_1^2(t)+e_2^2(t)+e_3^2(t))\right]^{\frac{p+q}{2p}}=\\
&-2k_hW(t)-2^{\frac{p+1}{2p}}\beta_hW^{\frac{p+q}{2p}}(t)=-2k_hW(t)-2^{\omega}\beta_hW^{\omega}(t),\forall t\ge 0
\end{aligned}, \tag{13}
$$

where $k_h=\min(k_1,k_2,k_3)$, $\omega=\dfrac{p+q}{2p}$, $\beta_h=\min(\beta_1,\beta_2,\beta_3)$.

So the Equation (13) satisfies Lemma 1.

Remark 1: p and q are two positive odd integers, $p+q$ is even number, $p>q$, so $0<\omega<1$. The reason for this is that if p is even and q is odd, then $e_i^{\frac{q}{p}}(t)$, $i\in\{1,2,3\}$ are always positive. Thus, it may cause the system to be unstable when $e_i(t)$ are negative. In contrast, if q is odd and p is even, then the states of $e_i^{\frac{q}{p}}(t)$, $i\in\{1,2,3\}$ will become complex numbers when $e_i(t)$ is negative.

Remark 2: In Equation (9), $e_1(t)$, $e_2(t)$ and $e_3(t)$ all choose the same exponent $\dfrac{p}{q}$. The reason for this is that if $e_1(t)$, $e_2(t)$ and $e_3(t)$ all choose the different exponent then we cannot apply Lemma 2 and Lemma 3 to satisfy conditions of Lemma 1.

From Equation (13), it is easy to deduce that

$$
\begin{aligned}
&\exp(2\alpha t)\times\dot{W}(t)+\exp(2\alpha t)\times 2\alpha W(t)=\\
&\frac{d}{dt}\left[\exp(2\alpha t)\times W(t)\right]\le 0,\qquad \forall t\ge 0.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\int_0^t\frac{d}{dt}\left[\exp(2\alpha t)\times W(t)\right]dt=\exp(2\alpha t)\times W(t)-W(0)\le\\
&\int_0^t 0dt=0,\qquad \forall t\ge 0.
\end{aligned} \tag{14}
$$

Thus, from Equations (11) and (14), it can be obtained that

$$\|e(t)\|^2=2W(t)\le 2W(0)\exp(-2\alpha t),\qquad \forall t\ge 0.$$

Consequently, we conclude that

$$\|e(t)\|\le\sqrt{2W(0)}\exp(-\alpha t),\qquad \forall t\ge 0$$

According to the results discussed above, synchronization is still achieved exponentially with the controller (9).

In this section, the main result is illustrated to show validity and correctness. Figure 1 - Figure 4 shows the numerical simulation results when the initial condition as follows,

$$x_1(0)=0.5,\ x_2(0)=0.1,\ x_3(0)=3,$$

$$y_1(0)=1,\ y_2(0)=1,\ y_3(0)=2,\ \omega=\frac{3}{5}.$$

Figure 1 shows the numerical simulation result when the initial parameters as follows,

$$k_h=k_1=k_2=k_3=1300,\ \beta_h=\beta_1=\beta_2=\beta_3=60.$$

Figure 2 shows the numerical simulation result when the initial parameters as follows,

$$k_h=k_1=k_2=k_3=1100,\ \beta_h=\beta_1=\beta_2=\beta_3=60$$

Figure 3 shows the numerical simulation result when the initial parameters as follows,

$$k_h=k_1=k_2=k_3=1000,\ \beta_h=\beta_1=\beta_2=\beta_3=50.$$

Figure 4 shows the numerical simulation result when the initial parameters as follows,

$$k_h=k_1=k_2=k_3=1000,\ \beta_h=\beta_1=\beta_2=\beta_3=90.$$

From Lemma 1 it can be obtained that the slave system (2) will synchronize the master system (1) within the time 3.5×10^{-3}, 4.35×10^{-3}, 4.95×10^{-3}, 4.15×10^{-3}, respectively. So it can be obtained that when β_h is fixed, the synchronization time is more short with the increasing of value of k_h. When k_h is fixed, the synchronization time is more short with the increasing of value of β_h.

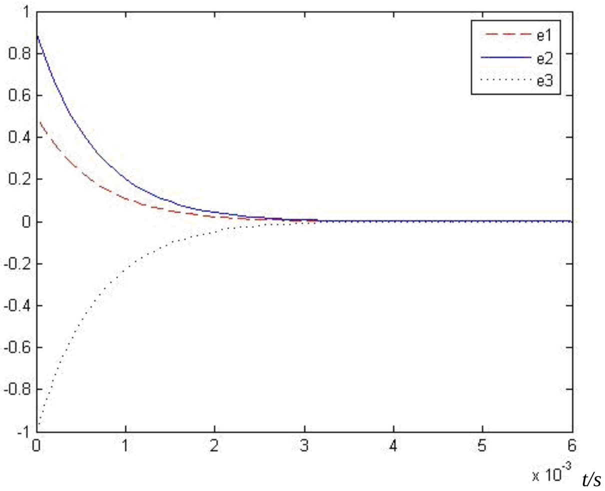

FIGURE 1 Synchronization error when $k_h = 1300, \beta_h = 60$

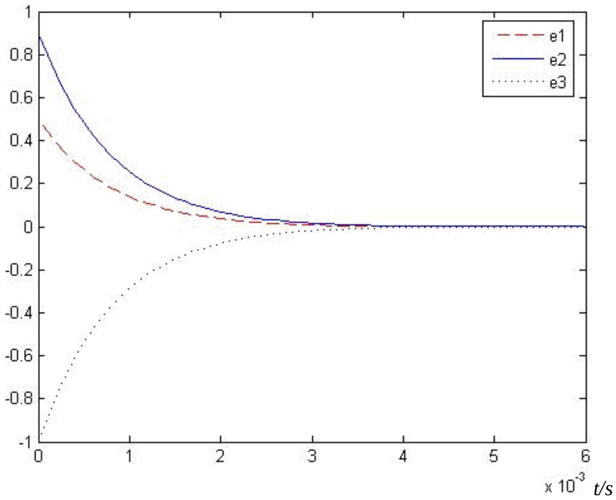

FIGURE 2 Synchronization error when $k_h = 1100, \beta_h = 60$

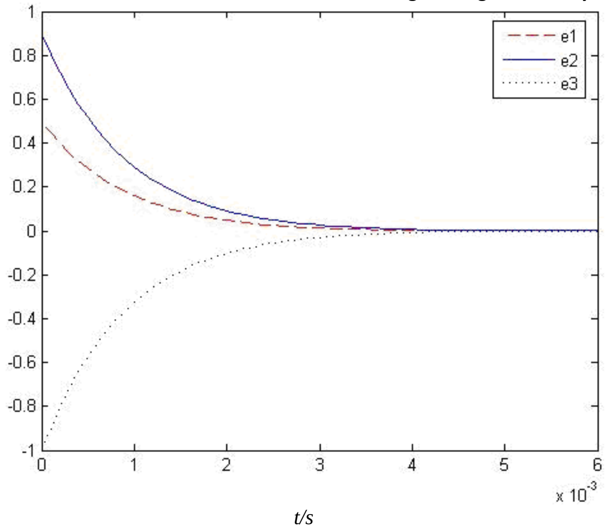

FIGURE 3 Synchronization error when $k_h = 1000, \beta_h = 50$

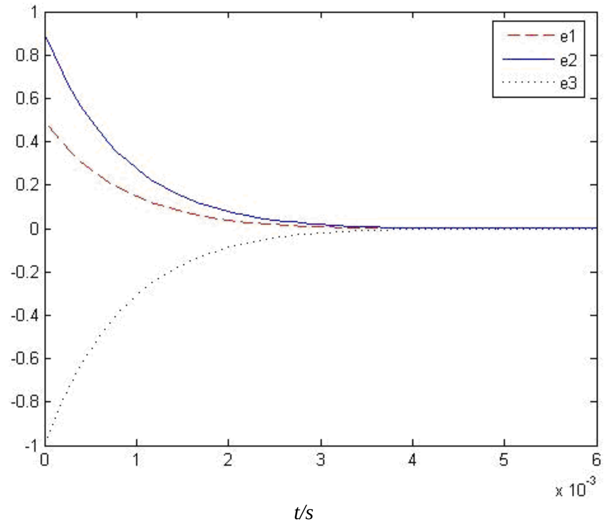

FIGURE 4 Synchronization error when $k_h = 1000, \beta_h = 90$

4 Conclusions

This paper proposes finite-time synchronization for the unified chaotic system. At the same time, the state of the slave system exponentially synchronizes the state of the master system. Based on the Lyapunov stability theory and the finite-time stability theory, the finite-time controller is derived to make the state of two unified chaotic systems is finite-time synchronization. Numerical simulations are presented to shows the effectiveness of theoretical analysis.

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