Research of numerical solutions of differential equations model based on the finite element method

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Abstract

Using the finite element method solving a class of second order ordinary differential equations, analyses the two-point boundary value problem of a class of second order ordinary differential equations, through numerical examples to validate its effectiveness.

Keywords: ordinary differential equations, finite element method, two-point boundary value problem

1 Introduction

Finite element method (fem) is a calculating method that booming in the 1960's, it has a wide range of applications, such as elasticity related problem [1-2], related problems in fluid mechanics [3], the heat conduction problem [4-5].

Using the finite element method solves the following two-point boundary value problem of second order ordinary differential equations:

\[ L_u := \frac{d}{dx}(p(x)\frac{du}{dx}) + r(x)\frac{du}{dx} + q(x)u = f(x) \quad a < x < b \quad (1) \]

\[ u(a) = u(b) = 0 \quad (2) \]

Among them \( p(x) \in C[a,b], r(x), q(x), f(x) \in C[a,b], p(x) \geq p_{\text{min}} > 0, q(x) \geq 0. \)

2 Generalized solution

Through the following method, the equation (3) and equation (1) are equivalent, the solution of original problem is called the classical solution, and solution meet the integral form is called the generalized solution. To determine the generalized solution of the new equation is the starting point of the finite element method (fem).

Use it multiply equation (1) both ends, and find the \( x \) integrate on \([a, b]\), using part of the integral, then we can get:

\[ -p(x)u'(x)\bigg|_x=a^x=b + \int_a^b p(x)u'(x)\phi'(x)dx + \int_a^b r(x)u'(x)\phi(x)dx \]

\[ + \int_a^b q(x)u(x)\phi(x)dx = \int_a^b f(x)\phi(x)dx \]

By the boundary conditions (2) and \( \phi(a) = 0 \), then

\[ \int_a^b p(x)u'(x)\phi'(x)dx + \int_a^b r(x)u'(x)\phi(x)dx + \int_a^b q(x)u(x)\phi(x)dx \cdot (3) \]

Then the \( u(x) \) is the general solution of boundary problem (1)-(2).

3 Element subdivision and interpolation

Let \( a = x_0 < x_1 < x_2 < \cdots < x_n = b \), element \( e_i \) is region \([x_i, x_{i+1}]\), let \( u(x) = u_i \) in \( x = x_i \), \( u_0 \) is known, \( u_1, \cdots, u_n \) is unknown. In finite element method (fem), when determining the subdivision, determine the specific form of interpolation polynomial on each small unit, and express it through the node function value, that means, in every element \( e_i \), let \( u(x) = N_i(x)u_i + M_i(x)u_{i+1} = [N]\{\delta\} \phi(x) = [N]\{\delta\}_i \), and \( N_i(x) = \frac{1}{L_i}(x_{i+1} - x_i), M_i(x) = \frac{1}{L_i}(x - x_i) \) is linear interpolation function.

\( L_i = x_{i+1} - x_i, \phi(x) \) is virtual displacement \([N] = (N_i(x), M_i(x)), \{\delta\}_i = (u_i, u_{i+1})^T \) is nodal displacement in element \( e_i \), \( \{\delta\}_i = (u_i', u_{i+1}')^T \) is node virtual displacement vector in element \( e_i \).

4 Computing units stiffness matrix and load vector

In this paper, the finite element method (fem) is given by specific physical instance and in the engineering, matrix \([K]\) reflects the unit’s rigidity, \([K]_{e_i} \cdot \{\delta\}_i = \begin{bmatrix} -U'_i \\ -U'_{i+1} \end{bmatrix} \)

show that in order to maintain deformation of the unit \( e_i \), two endpoints nodes \( x = x_i, x_{i+1} \) in unit \( e_i \) need external force, to his balance, the two external force is the force

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$-U_i^e$ and $-U_{ij}^e$ through node, and the force known as the equivalent nodal force. While unit load vector $\{F\}_i$ reflects the effect on the unit $e_i$ displacement distribution of physical strength, its each component shows after the

$$\sum_{i=1}^{n+1} \int \left[ p(x)u'(x)\phi'(x) + r(x)u'(x)\phi(x) + q(x)u(x)\phi(x) \right] dx = \sum_{i=1}^{n+1} \int f(x)\phi(x) dx .$$

(4)

Using $u, u_{i+1}$ and $u_i, u_i'$ respectively represent

$$\int (pu'\phi' + ru'\phi + q\phi) dx. \int f dx.$$ If a number of a is regarded as a matrix of order, we can see $a^T = a$, then

$$\int (pu'\phi' + ru'\phi + q\phi) dx = \int p(B)^T (B) [N]^T (N) dx + \int r(N) [N] (B)^T (B) [N] dx + \int q(N) [N] (N) dx \int \delta \delta_d dx .$$

(5)

$[K]_i$ is called element stiffness matrix, its concrete form is:

$$[K]_i = \int_{x_i}^{x_{i+1}} (p(B)^T [B] + [N]^T [B] + q(N)^T [N]) dx$$

$$= \left[ \begin{array}{cc} \frac{P_i}{L_i} + \frac{r_i}{L_i} \int_{x_i}^{x_{i+1}} N_i dx + \int_{x_i}^{x_{i+1}} qN_i^2 dx & -\frac{P_i}{L_i} + \frac{r_i}{L_i} \int_{x_i}^{x_{i+1}} N_i dx + \int_{x_i}^{x_{i+1}} qN_i M_i dx \\ -\frac{P_i}{L_i} + \frac{r_i}{L_i} \int_{x_i}^{x_{i+1}} N_i dx + \int_{x_i}^{x_{i+1}} qN_i M_i dx & \frac{P_i}{L_i} + \frac{r_i}{L_i} \int_{x_i}^{x_{i+1}} N_i dx + \int_{x_i}^{x_{i+1}} qN_i^2 M_i dx \end{array} \right] .$$

Among it, $p_i = \int_{x_i}^{x_{i+1}} p(x) dx$.

Make the element stiffness matrix as

$$[K]_i = \begin{pmatrix} k_{ii} & k_{ij} \\ k_{ji} & k_{jj} \end{pmatrix},$$

then stiffness coefficient $k_{ij}$ is

$$k_{ii} = \frac{P_i}{L_i} + \frac{r_i}{L_i} \int_{x_i}^{x_{i+1}} N_i dx + \int_{x_i}^{x_{i+1}} qN_i dx$$

$$k_{ij} = k_{ji} = \frac{P_i}{L_i} + \frac{r_i}{L_i} \int_{x_i}^{x_{i+1}} N_i dx + \int_{x_i}^{x_{i+1}} qN_i M_i dx,$$

$$k_{ji+1} = k_{ij+1} = \frac{P_i}{L_i} + \frac{r_i}{L_i} \int_{x_i}^{x_{i+1}} N_i dx + \int_{x_i}^{x_{i+1}} qN_i dx .$$

(6)

Expressed $\{F\}_i$ as $\{F\}_i = \left[ F_{i+1} \right] [F]_i$, then unit load coefficient is $F_{i+1} = \int_{x_i}^{x_{i+1}} N_i f dx, F_{i+1} = \int_{x_i}^{x_{i+1}} M_i f dx$.

5 Total stiffness matrix and the total load vector

Expand $[K]_i, \{F\}_i, \{\delta\}_i$ and $\{\delta^*\}$ for $n + 1$ order matrix and the $n + 1$ d vector, put (5) and (6) into (4), then

$$\{\delta^*\}^T (\sum_{i=1}^{n+1} [K]_i \{\delta\}_i = \{\delta^*\}^T (\sum_{i=1}^{n+1} \{F\}_i) .$$

(7)

Among them, $[K] = \sum_{i=1}^{n+1} [K]_i, \{F\} = \sum_{i=1}^{n+1} \{F\}_i$ as the total stiffness matrix and the total load vector respectively.

6 Constraint handling

As on the endpoint $x = 0, u(x)$ satisfy the boundary conditions (2), therefore $u_0 = 0$, in addition $\phi(0) = 0$, so
coordinates of nodes are

\[ x_0 = 0, \ x_1 = \frac{1}{4}, \ x_2 = \frac{1}{2}, \ x_3 = \frac{3}{4}, \ x_4 = 1 \]

Stiffness matrix:

\[
\begin{bmatrix}
K_{ii}
\end{bmatrix} = \begin{bmatrix}
4(x_{i+1} - x_{i+1}x_i + x_i^2) - \frac{20}{3} x_i x_{i+1} \\
\frac{4}{3}(x_{i+1} + x_i^2) - \frac{20}{3} x_i x_{i+1} \\
\frac{4}{3}(x_{i+1} + x_i^2) - \frac{20}{3} x_i x_{i+1}
\end{bmatrix}
\]

Unit load vector:

\[
\begin{bmatrix}
F_i
\end{bmatrix} = \begin{bmatrix}
2(x_{i+1}x_{i+1} + x_i x_{i+1} + x_i^2)
- \frac{1}{2}(x_{i+1} + x_i)(x_i^2 + x_i^2)
\frac{1}{2}(x_{i+1} + x_i)(x_i^2 + x_i^2) - \frac{2}{3} x_{i+1}(x_{i+1} + x_i x_i^2)
\end{bmatrix}
\]

The overall stiffness matrix is \( [K] = \sum_{i=0}^{3} [K]_{ii} \), total load vector is \( [F] = \sum_{i=0}^{3} [F]_{i} \). The exact solution is \( u(x) = x - \frac{1}{2} x^2 \), the numerical results are shown in Table 1.

### Table 1 Numerical result

<table>
<thead>
<tr>
<th>node</th>
<th>Exact solution</th>
<th>Approximate solution</th>
<th>Absolute error</th>
<th>Relative error</th>
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<tbody>
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<td>0.21637</td>
<td>2.3E-03</td>
<td>1.08E-02</td>
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</tr>
</tbody>
</table>

This paper discusses the application of finite element method for solving a class of second order ordinary differential equation numerical solution, analyses a class of second order ordinary differential equation with two-point boundary value problem, the effectiveness of finite element method is verified by a numerical example for solving a class of second order ordinary differential equation numerical solutions algorithm.

### References


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