Synchronization optimization under symmetry network

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Abstract

Rich symmetries have been found in many real networks, which is an extensive structural of networks. We study the relationships between synchronization and symmetry of network. One of fascinating problems related to symmetry network is how to optimization phase synchronization by employing symmetric structure. For this purpose, we optimize the network structure by symmetry properties of network, which is statistics the orbits of network. Nodes in the same orbit have similar properties, and then we reduce the network size by nodes in same orbit. We simulate the BA and SW model. The simulations result show that symmetry structure can optimize the network topology and can enhance phase synchronization of network.

Keywords: synchronization, symmetry network, orbit

1 Introduction

The topologies of network have been extensively studied and some common architecture has been discovered [1-3]. The small-world property [4-6], the scale-free [7], communication structure [7-9], network motif [10], modular network [11] any symmetric network [12, 13]. From this viewpoint, systematically understanding the network structural effects on their dynamical processes is of both theoretical and practical importance. Synchronization behaviour, in particular, as a widely observed phenomenon in network systems, has received a great deal of attention in the past few decades [14, 15]. The effects of average distance [16, 17], heterogeneity [14, 18], clustering [19, 20] and weight distribution on network synchronizability have been extensively investigated [21, 22]. Since the synchronizability is correlated with many topological properties, but these properties affected the synchronization maybe is not a proper answer. Perhaps a good index, which can characterize the synchronizability, has not been found. In recent years, graph-theoretic were used to analyse the network synchronizability, e.g., degree sequences were discussed [23], complementary graph and subgraph techniques were used [24-26]. But using the automorphism of symmetry graph to study the synchronization has not been fully investigated. Motivated by the above of work, this paper focuses on the relationships between the network synchronization and the orbit of automorphism of symmetry network. We know that it is difficulty all nodes achieves the synchronization in whole network, the finding shows that the network size is smaller and achieving the synchronization is easier. From this, we know that we may optimize the network topology to enhance the network synchronizing through symmetric structure. In symmetry network nodes in same orbit are similar and we view the nodes in same orbit as one node and reduce the network size. We found that the nodes in same orbit are also very easy to achieve the synchronization. The results show that the optimized networks enhance the network synchronism.

2 Symmetry network and synchronization

2.1 SYMMETRY NETWORK

A graph or network is denoted by $H=(V,E)$, where V is the set of vertices and E is the set of edges. To any node $v\in V(G)$, $\varepsilon(v)=v$, obviously, there has a isomorphism to identity map $\varepsilon: V(H)\to V(H)$ [28]. We illustrate the concept through Figure 1. There has a map $\alpha_1: V(H)\to V(H)$ defined by $\alpha_1(v)=v$, which is a isomorphism from H to itself [27-29].

An automorphisms of a graph H is a isomorphism from H to itself. We denote the set of automorphisms of graph H under composition operation as $Aut(H)$. Figure 1 shows $Aut(H)=\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}$. The automorphisms group also is a permutation group. The set of all of permutation group base on n called symmetric group S_n. The network is considered as symmetry if its underlying graph contains not only an identity permutation.

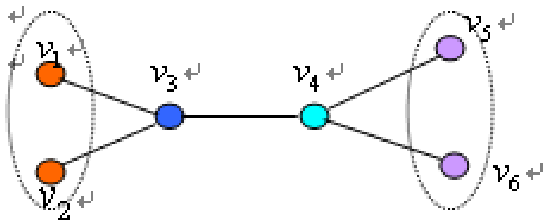

FIGURE 1 Symmetry network

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Next, we will introduce the concepts of automorphism partition and orbit. Given the automorphism group acting on vertex set V, we can get a partition $\pi=\{V_0,V_1,...,V_k\}$, $x\,R\,y$ if and only if $\exists\alpha\in Aut(H)$, s.t. $\alpha(x)=y$. Such partition is called automorphism partition, each cell of the partition is called an orbit of $Aut(H)$, denote as $orbit(H)$. An orbit is trivial if it only contains a single vertex, otherwise, the orbit is non-trivial. The vertex u,v called similar if they are in same orbit. Figure 1 shows the automorphism partition $\pi=\{\{v_1,v_2\},\{v_3\},\{v_4\},\{v_5,v_6\}\}$ and the orbits. Nodes in same orbit are marked with the same colour, shown as Figure 1. The nodes in same orbit have similar structure information [28]. We view them as one node, e.g.:

$\{v_1,v_2\}\to v_1,\{v_3\}\to v_2,\{v_4\}\to v_3,\{v_5,v_6\}\to v_4$.

We reduce the network structure, shown as Figure 2. We call this reduction network as optimization network in this paper.

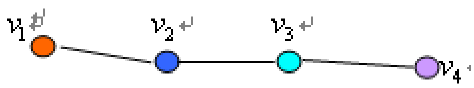


FIGURE 2 Optimization network.

2.2 SYNCHRONIZATION

To settled the natural frequencies of the oscillators, Kuramoto worked out a mathematically model to tackle this problem and each of the oscillators in the system are [30-31]:

$$\dot{\theta}_i=\omega_i+\frac{K}{N}\sum_{j=1}^{N}\sin\left(\theta_j-\theta_i\right),\ (i=1,...,N), \tag{1}$$

where N is the factor and $N\to\infty$, ω_i is the frequency of oscillator i, K is the coupling constant.

To deal with the synchronization on complex networks, the Kuramoto model can be reformulated to Equation (2): The synchronization of Kuramoto model [30, 31]:

$$\frac{d\phi_i}{dt}=\omega_i-\frac{\delta}{N}\sum_j a_{ij}\sin\left(\phi_i-\phi_j\right), \tag{2}$$

where ϕ_i and ω_i are the phase and the intrinsic frequency of vertex i, respectively. ω_i is chosen from the Gaussian distribution with unit variance. The summation runs over j, the nearest neighbours of vertex i. The coupling strength is used as δ. To study synchronization, each node in complex topologies is considered an oscillator, such as WS networks and BA graphs.

The order parameter M is defined as:

$$M=\left\langle\left|\frac{1}{N}\sum_{j=1}^{N}e^{i\phi j}\right|\right\rangle, \tag{3}$$

where $\langle...\rangle$ is the ensemble average over different configurations and N is the total number of vertices. M is $0(1)$ in the fully incoherent (coherent) phase.

The dynamics of synchronization on symmetry networks through a modified Kuramoto model, where $orbit[i]$ is the vertex i in the same orbit.

$$\frac{d\phi_i}{dt}=\omega_i-\frac{\delta}{Orbit[i]}\sum_j a_{ij}\sin\left(\phi_i-\phi_j\right), \tag{4}$$

3 Scale-free network

To scale-free network our numerical simulations are based on the SF model. The model starts from m_0 nodes, at every time step adds a new node with $m(\le m_0)$ edges that link the odd node already existing in network. We assume that the probability connecting to an existing node i is proportional to $i's$ degree k_i. The algorithm of the BA model is [35]:

1) Growth: starting with a small number m_0 of nodes, each step adds a new node with $m(\le m_0)$ edges connect new node with different old nodes in network.

2) Preferential growth: when choosing old nodes which the new node connects to, and assuming that the probability depends on the degree k_i of node i. The link probability is:

$$\prod=\frac{k_i}{\sum_j k_j}. \tag{5}$$

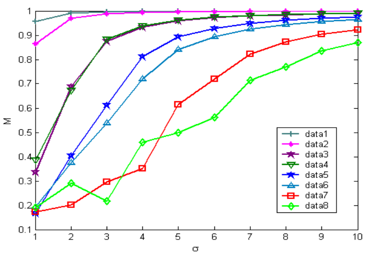
By using the rate equation, we can easily obtain the degree distribution of the whole network, $p(k)\sim k^{-2.8}$. In simulation, there are totally 200 nodes in the network. According to the new node m which added to the network, we assumed $m=2, m=3, m=4, m=5$, respectively. When $m=2$, the $Aut(m_2)\approx 6.8\times10^6$, so the network is richly symmetric [10]. We only simulated the synchronization of non-trivial, because the nodes in non-trivial at lease are 2. In the 11th and 17th orbit, the node number is 14, respectively. In the 16th orbit, the node number is 17. In the 20th orbit, the node number is 8. In the 25th orbit, the node number is 9. In the 37th and 38th orbit, the node number is 3, respectively. In other orbit, the node numbers also are 2. To get the synchronization property of network, we investigate how the order parameters in different orbits, a network and quotient network change with the coupling strength. According to the Equation (2), Figure 3a shows the simulation results. From top to bottom, the node numbers in different orbits 2, 3, 8, 9, 14, and 17, corresponding to more and more node numbers in same orbit. In the case $orbit[i]=2$, with the increasing of coupling strength, the order parameter of

$orbit[i]$ soon reaches 1, indicating the synchronized state within one orbit. With the further observed, an interesting phenomenon appears: for different $orbit[i]=2,3,8,9,14,17$, order parameter for different $orbit[i]$ all reach 1 soon, which indicated oscillators belong to the different $orbit[i]$ are synchronized. We call this phenomenon orbit synchronization clustering.

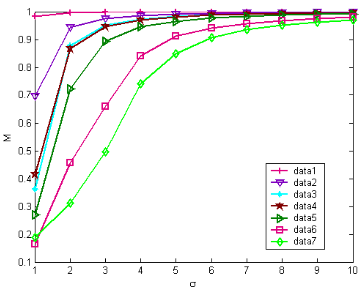
However, the phases of oscillators in different $orbit[i]$ are different, in the case $orbit[i]=17$, the order parameter of $orbit[i]$ reaches 1 slower than $orbit[i]=2$, which indicating more node within one orbit, slower reached synchronized state.

Similarly, to enhance the synchronization of the whole network, we defined the node in the same orbit as one node, like this may reduce the node number in the network greatly and enhances the network synchronism. Synchronization of the optimized network and original network shown as Figure 3a, the red square represents optimized network, indicated as $data8$, the green diamond represents the whole network, indicated as $data9$. In Figure 3a, obviously, the order parameter for the whole network is the lowest one. Optimization network reach 1 sooner than the whole network. That is, optimized network's topology may enhance the network synchronized performance.

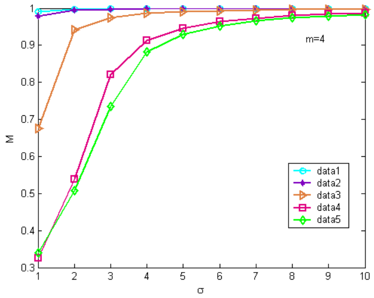
At the same time, we simulated the synchronism of $orbit[i]$, the optimized network and the whole network with $m=3, m=4, m=5$, shown in Figures 3b-d. The simulation show that different m values form different network structure, but each network $orbit[i]$ may achieve the synchronization and forms the synchronized clustering. The synchronization of optimization network with different m values surpasses the original network.



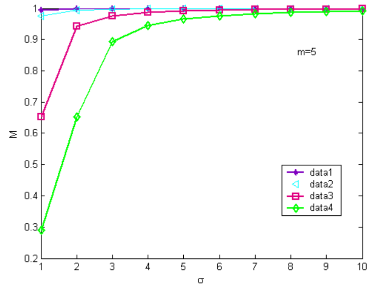
a) m=2



b) m=3



c) m=4



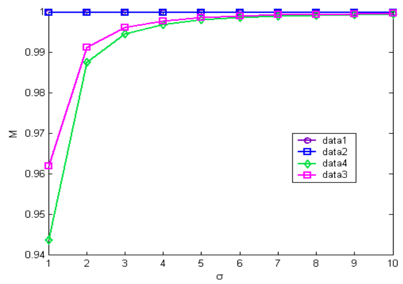
d) m=5

FIGURE 3 Synchronization of BA network

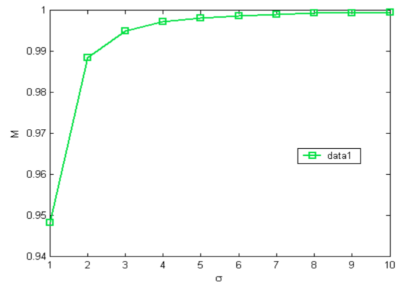
4 Small world network

The WS model was proposed by Watts and Strogatz. The WS model start from regular loop network with n nodes. Assuming that each node on loop links K edges with two sides nodes on it, then restart a new link with the probability p to each edge. This new network has small world character, which has smaller average path and bigger cluster. The degree distribution of small world obey the Possion distribution.

We made the synchronization simulation of small world network similar to BA network. Network has 200 nodes in simulation same as BA network size. When network connection probability smaller than $p=0.49$, network has not non-trivial orbit, only trivial orbit. Hence, assuming that WS network connection probability is $p=0.49$ and $p=0.5$ separately. When the connection probability is bigger than $p=0.5$, all of nodes in the same orbit. When $p=0.49$, $Aut(H)$ is very small and has a few non-trivial orbit. We simulated the synchronization of non-trivial orbit and the optimization network when $p=0.49$, shown as Figure 4a. When $p\geq 0.5$, $Aut(H)\approx 7\times 10^5$ the network has richly symmetry and all of the node in same orbit. Although the topology of network has not optimized, the whole network reached 1 very soon, that is the network has better synchronization, shown as Figure 4b.



a)



b)

FIGURE 4 Synchronization of SW network

We found that the synchronization of SW reaches 1 sooner than BA model with the same scale. In BA, the network symmetry is richer, different non-trivial orbit are more and enhance the network synchronism. In SW, The network symmetry is richly, the node in the same orbit and the network is very completely easy to achieve the synchronization.

5 Conclusion

The paper captured the synchronization of symmetric network and introduced the concept of symmetry network. The more automorphism of a network have and the more symmetry of a network and the orbit constitutes the partition of automorphism. On the other hand, the nodes in orbit have similar properties, hence, we viewed the nodes in orbit as a node to optimize the network topology, and we called this network as optimization network. We simulated the synchronization of orbit clustering, the whole network and the optimization network using revised Kuramoto model. We discussed the relationship of synchronization and symmetry of BA network. In BA model, the different m values and the network topology are also different. According the different network topology formed by different m values, we simulated the synchronization of orbit clustering, the whole network and the optimization network of BA model. The results show that the optimized network topology can enhance the synchronization of network. Finally, we discussed the SW model, the synchronization of network is different according to different linking probability. All node in the orbit when $p \geq 0.5$, the network has the better synchronization. In BA model, the nodes in different orbit can reduce the network scale to optimize the network topology, and the optimization network has the better synchronization than the original network. To SW model, the nodes of the network in the same orbit when $p \geq 0.5$, and it can't optimize the network structure. But the results show that the SW model is more easy reach the synchronization than BA model to the same network scale if BA model is not optimization.

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