Global convergence of a predictor-corrector smoothing newton method for generalized nonlinear complementarity problem

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Abstract

A predictor-corrector smoothing Newton method for the generalized nonlinear complementarity problem is proposed based on a class of smoothing functions. Dislike the complementarity problem, there were more functions needed to be considered in generalized nonlinear complementarity problem (GNCP). Therefore, the first thing was to reformulate the GNCP to a system of smoothing equations. Then a predictor-corrector smoothing Newton algorithm was modified to solve the problem. Under some suitable conditions, the boundness of the iteration sequence generated by the proposed smoothing Newton method is proved. To describe the global convergent properties of the method, it shows that any accumulation point of the generated sequence is a solution of the generalized nonlinear complementarity problem.

Keywords: generalized complementarity problem, smoothing function, smoothing Newton method, global convergence

1 Introduction

Consider the generalized nonlinear complementarity problem (denoted by GNCP), which is to find a vector $x \in R^n$ such that

$$s = f(x) \geq 0 \qquad t = g(x) \geq 0 \quad s^{\mathrm{T}} t = 0, \tag{1}$$

where $f, g : R^n \to R^n$ are continuously differentiable mappings. GNCP finds important applications in many fields, such as engineering and economics, and is a wide class of problems that contains the classical nonlinear complementarity problem (abbreviated as NCP). For example, $g(x) = x$, then GNCP reduces to the nonlinear complementarity problem. Moreover, when $f(x) = Mx + q$, $M \in R^{n \times n}$, $q \in R^n$, NCP becomes a linear complementarity problem (LCP).

The nonlinear complementarity problem has attracted much attention due to its various applications. We refer the interested readers to the survey papers [1, 2] and references therein. Many numerical methods for solving NCP have been developed [3-8]. It is well known that the smoothing Newton method received much attention in solving NCP due to its high efficiency [6, 9-12]. The basic idea of smoothing Newton-type methods is to employ a smoothing function to reformulate the problem concerned as a system of smoothing equations and then to solve the smoothing equations approximately by using Newton's method at each iteration. By making the parameter to tend to zero, A solution of the original problem is obtained. To solve the GNCP, it is often reformulated as a minimization problem over a simple set or an unconstrained problem, see [13] for the case that κ is a general cone and see [14] and [15] for the case that $\kappa = R^n_+$. The conditions under which a stationary point of the reformulated optimization is a solution of the GNCP were provided in this literature. Predictor-corrector smoothing Newton method has been proposed for solving NCP [16]. In this paper, the smoothing Newton method for NCP is extended to solve the generalized nonlinear complementarity problem based on a class of smoothing functions. Motivated by the above discussions, a predictor-corrector smoothing Newton algorithm is proposed for solving GNCP. Based on a class of smoothing functions, the smoothing functions have been proved to possess good properties. GNCP is reformulated to a system of smoothing equations. Predictor-corrector smoothing Newton algorithm can be used to solve the smoothing equations for the solution of the GNCP. Under suitable conditions, we show that any accumulation point of the generated sequence is a solution of the GNCP, and we also establish the global convergence under weaker conditions.

The following part of the paper is organized as follows. In Section 2, we present some preliminary results and a class of smoothing functions. In Section 3, we give a predictor-corrector smoothing Newton algorithm. The global convergence of the algorithm is discussed in Section 4. The conclusions are given in Section 5.

The following notions will be used throughout this paper. All vectors are column vectors. The subscript T denotes transpose. R^n denotes the space of real column vectors. R^n_+ and R^n_{++} denote the nonnegative and positive orthants of R^n respectively. Let $I = \{1, 2, \cdots, \mathrm{n}\}$. For any $u \in R^n$, $diag\{u_i, i \in N\}$ denotes the diagonal matrix

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whose i[th] diagonal element is u_i and $vec\{u_i, i \in N\}$ denotes the vector u. The symbol $\|\cdot\|$ stands for the 2-norm. For any vector $s, t, x \in R^n$, we write $\left(s^{\mathrm{T}}, t^{\mathrm{T}}, x^{\mathrm{T}}\right)^{\mathrm{T}}$ as (s,t,x) for simplicity. For any (u,s,t,x), $\left(u_k, s^k, t^k, x^k\right) \in R_+ \times R^{3n}$, we always use the following notations throughout this paper unless stated otherwise: $w := (s,t,x)$, $w^k := \left(s^k, t^k, x^k\right)$, $z := (u,w)$ $z^k := \left(u_k, w^k\right) := \left(u_k, s^k, t^k, x^k\right)$

2 Preliminaries

First, we review some useful definitions and results used in the subsequent.

2.1 DEFINITION

A function $f : R^n \to R^n$ is said to be a p_0-function, if for all $x, y \in R^n$ with $x \neq y$, there exists an index $i_0 \in I$ such that $x_{i_0} \neq y_{i_0}, \left(x_{i_0} - y_{i_0}\right)\left(f\left(x_{i_0}\right) - f\left(y_{i_0}\right)\right) \geq 0$.

2.2 DEFINITION

A matrix $M \in R^{n \times n}$ is said to be a p_0-matrix, if all its principal minors are non-negative.

2.3 DEFINITION

A function $\phi : R^2 \to R$ is called an NCP function if it satisfies $\phi(a,b) = 0 \Leftrightarrow a \geq 0, b \geq 0, ab = 0$

Over the past decades, a variety of NCP functions have been studied, refer to [17, 18] and references therein. Among which, a popular NCP function intensively studied recently is the well-known Fischer-Burmeister NCP function [19, 20] defined as

$$\phi(a,b) = \sqrt{a^2 + b^2} - (a+b). \tag{2}$$

In this paper, we propose and investigate a family of NCP functions based on the Fischer-Burmeister function. $\phi_p : R^3 \to R$

$$\phi_p(u,a,b) = a + b - \sqrt[p]{|a|^p + |b|^p + u}, \tag{3}$$

where p is any fixed real number in the interval $(1, +\infty)$.

2.4 LEMMA

$$\phi_p(0,a,b) = 0 \Leftrightarrow a \geq 0, b \geq 0, ab = 0$$

2.5 LEMMA

Let $\hat{\phi}_p = a + b - \sqrt[p]{|a|^p + |b|^p}$ where $p > 1$ .If $\left\{\left(a^k, b^k\right)\right\} \subseteq R^2$ with $\left(a^k \to -\infty\right)$ or $\left(b^k \to -\infty\right)$ and $\left(a^k \to \infty \textit{ and } b^k \to \infty\right)$, then we have $\left|\hat{\phi}_p\left(a^k, b^k\right)\right| \to \infty$ for $k \to \infty$.

It is easy to see that the Lemma 2.5 is true from [21]. Define

$$\Phi_p(u,s,t) = \begin{bmatrix} \phi_p(u, s_1, t_1) \\ \vdots \\ \phi_p(u, s_n, t_n) \end{bmatrix}, \tag{4}$$

where $(u,s,t) \in R \times R^n \times R^n$. Then we can define a reformulation function of GNCP, $H : R^{3n+1} \to R^{3n+1}$ by

$$H(z) := \begin{pmatrix} u \\ s - f(x) \\ t - g(x) \\ \Phi_p(z) \end{pmatrix}, \tag{5}$$

where $z = (u,s,t,x) \in R \times R^n \times R^n \times R^n$. Thus, by Lemma 2.4, it is easy to see that $H(u,s,t,x) = 0 \Leftrightarrow$ x is a solution of GNCP(1).

It is not difficult to see that the function $H(z)$ is continuously differentiable if f and g in GNCP (1) are continuously differentiable for any $(u,s,t,x) \in R_{++} \times R^n \times R^n \times R^n$

2.6 LEMMA

Let M be a p_0 matrix and any positive diagonal matrix D_1 and D_2, then the matrix $D_1 + D_2 M$ is non-singular.

2.7 LEMMA

Let $H := R^{3n+1} \to R^{3n+1}$ and $\Phi_p := R^{2n+1} \to R^n$ be defined by (5) and (4), respectively. Then we have

(a) Φ_p is continuously differentiable at any $(u,s,t) \in R^{n+1} \times R^n$ with $u \neq 0$.

(b) H is continuously differentiable any $z := (u,s,t,x) \in R_{++} \times R^n \times R^n \times R^n$ with its Jacobian

$$H'(z)=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & E & 0 & -f'(x) \\ 0 & 0 & E & -g'(x) \\ d(z) & D(z) & E(z) & 0 \end{pmatrix}, \qquad (6)$$

where

$$d(z):=vec\left\{\phi_p'(u,s_i,t_i):i\in I\right\}, \qquad (7)$$

$$D(z):=diag\left\{1-\frac{sign(s_i)|s_i|^{p-1}}{\left(\sqrt[p]{|s_i|^p+|t_i|^p+u}\right)^{p-1}}:i\in I\right\}, \qquad (8)$$

$$E(z):=diag\left\{1-\frac{sign(t_i)|t_i|^{p-1}}{\left(\sqrt[p]{|s_i|^p+|t_i|^p+u}\right)^{p-1}}:i\in I\right\}, \qquad (9)$$

$D(z)$ and $E(z)$ are positively definite if $u>0$.

3 Algorithm

In this section, we formally present our predictor-corrector smoothing Newton algorithm for solving $H(z)=0$ by using the smoothing function $\phi_p:R^3\to R$.

Algorithm 3.1

Step 0. Initialization. Choose $\delta,\gamma,\sigma\in(0,1)$, $p\in(1,+\infty)$. Taking an arbitrary vector $z^0:=(u_0,s^0,t^0,x^0)$ $u_0>0$, $(s^0,t^0,x^0)\in R^n\times R^n\times R^n$, choose $\beta>1$ such that $\left\|H(z^0)\right\|\le\beta u^0$. Set $e^0:=(1,0,\cdots,0)\in R^{3n+1}$ and $k:=0$.

Step 1. Termination Criterion. If $\left\|H(z^k)\right\|=0$, stop.

Step 2. Predictor Step. If $\left\|H(z^k)\right\|>1$,set $\hat{z}^k:=z^k$ and go to Step 3; otherwise, computer a perturbed Newton direction $\Delta\hat{z}^k:=\left(\Delta u^k,\Delta\hat{s}^k,\Delta\hat{t}^k,\Delta x^k\right)\in R^{3n+1}$ by the linear system

$$H'(z^k)\Delta\hat{z}^k=-H(z^k)+\left(1/\beta\right)\left\|H(z^k)\right\|^{1+r}e^0. \qquad (10)$$

If

$$\left\|H\left(z^k+\Delta\hat{z}^k\right)\right\|\le\left\|H(z^k)\right\|^{1+r}. \qquad (11)$$

Then let $\hat{z}^k:=z^k+\Delta\hat{z}^k$; otherwise, let $\hat{z}^k:=z^k$.

Step 3. Corrector Step. If $\left\|H\left(\hat{z}^k\right)\right\|=0$, stop; otherwise, computer a perturbed Newton direction $\Delta\tilde{z}^k:=\left(\Delta u^k,\Delta\tilde{s}^k,\Delta\tilde{t}^k,\Delta x^k\right)\in R^{3n+1}$ from the linear system

$$H'\left(\hat{z}^k\right)\Delta z^k=-H\left(\hat{z}^k\right)+\left(1/\beta\right)\left\|H\left(\hat{z}^k\right)\right\|e^0. \qquad (12)$$

Let θ^k be the maximum of the values $1,\delta,\delta^2,\cdots$ such that

$$\left\|H\left(\hat{z}^k+\theta^k\Delta\tilde{z}^k\right)\right\|\le\left[1-\sigma\left(1-1/\beta\right)\theta^k\right]\left\|H\left(\hat{z}^k\right)\right\|. \qquad (13)$$

Set $z^{k+1}:=\hat{z}^k+\theta^k\Delta\tilde{z}^k$.

Step 4. Update. Set $k:=k+1$ and go to Step 1.

The following theorems show that Algorithm 3.1 is well defined and infinite sequence is generated by the Algorithm with nice properties. Define the set $\Omega:=\left\{z=(u,s,t,x)\in R^{3n+1}:\left\|H(z)\right\|\le\beta u\right\}$ with $\beta>1$.

Now we show that the Eq. (10) and equation (12) are solvable under suitable conditions.

3.1 LEMMA

Let f and g in GNCP (1) be continuously differentiable. Assume that either $f'(x)$ is non-singular, $g'(x)f'(x)^{-1}$ is a p_0 matrix; or $g'(x)$ is non-singular, $f'(x)g'(x)^{-1}$ is a p_0 matrix, then if $u>0$, $H'(z)$ is non-singular.

Proof. Suppose that

$$H'(z)(\Delta u,\Delta s,\Delta t,\Delta x)=0. \qquad (14)$$

Then we need to prove that $\Delta z=0$, where $\Delta z=(\Delta u,\Delta s,\Delta t,\Delta x)$. From (6) and (14), we obtain

$$\begin{cases}\Delta u=0\\ \Delta s-f'(x)\Delta x=0\\ \Delta t-g'(x)\Delta x=0\\ d(z)\Delta u+D(z)\Delta s+E(z)\Delta t=0\end{cases}. \qquad (15)$$

By equation (15), we have

$$\left[D(z)f'(x)+E(z)g'(x)\right]\Delta x=0. \qquad (16)$$

If $f'(x)$ is non-singular, $g'(x)f'(x)^{-1}$ is a p_0 matrix, then equation (16) is equivalent to

$$\left[D(z)+E(z)g'(x)f'(x)^{-1}\right]f'(x)\Delta x=0. \qquad (17)$$

Based on $D(z)$ and $E(z)$ are positively definite and $g'(x)f'(x)^{-1}$ is a p_0 matrix, using Lemma 2.6, we derive that $D(z)+E(z)g'(x)f'(x)^{-1}$ is non-singular. Thus by equation (17) $\Delta x = 0$. Furthermore, by the second and third equation of Eq (15), $\Delta s = 0$ and $\Delta t = 0$. So we can obtain that $\Delta z = 0$. Hence, $H'(z)$ is non-singular on $R_{++} \times R^n \times R^n \times R^n$.

3.2 THEOREM

Let the sequence $\{z^k = (u_k, s^k, t^k, x^k)\}$ be generated by Algorithm 3.1, where $u_k > 0$. Suppose that the conditions of Lemma 3.1 are satisfied Then Algorithm 3.1 is well defined.

Proof. Suppose that we are now in the position of the k th iterate; we prove that the $(k+1)$ th iterate of the algorithm is well defined. If $H(z^k) = 0$, by a simple continuity discussion, we establish that $\{z^k\}$ is a solution of GNCP. Thus, we suppose that $\|H(z^k)\| > 0$. We consider the following two cases, namely: Case (i), (ii) below.

Case (i) By Lemma 3.1, it follows that the matrix $H'(z^k)$ and $H'(\hat{z}^k)$ are non-singular, which demonstrate that (10) and (12) are well defined.

Case (ii) Given $k \ge 0$, for any $\alpha \in (0,1]$, let

$$r(\alpha) := H\left(\hat{z}^k + \alpha \Delta \tilde{z}^k\right) - H\left(\hat{z}^k\right) - \alpha H'\left(\hat{z}^k\right) \Delta \tilde{z}^k. \quad (18)$$

Then by (12) and (18),

$$\left\|H\left(\hat{z}^k + \alpha \Delta \tilde{z}^k\right)\right\| = \left\|(1-\alpha) H\left(\hat{z}^k\right) + \left(\alpha/\beta\right)\left\|H\left(\hat{z}^k\right)\right\| e^0 + r(a)\right\|$$

$$\le \left[1 - \alpha\left(1 - 1/\beta\right)\right]\left\|H\left(\hat{z}^k\right)\right\| + \|r(\alpha)\|. \quad (19)$$

By noting that $f(x)$ and $g(x)$ are continuously differentiable for any $x \in R^n$ using Lemma 2.7. and $u^k > 0$ for any $k \ge 0$, we know that $H(z)$ is continuously differentiable at $\hat{z}^k$, which yields $\|r(\alpha)\| = o(\alpha)$ by (18). Therefore, from (19), it follows that there exists an $\bar{\alpha} \in (0,1]$ such that the inequality

$$\left\|H\left(\hat{z}^k + \alpha \Delta \tilde{z}^k\right)\right\| \le \left[1 - \sigma\left(1 - 1/\beta\right)\alpha\right]\left\|H\left(\hat{z}^k\right)\right\|, \quad (20)$$

holds for any $\alpha \in (0, \bar{\alpha}]$. This demonstrates that (13) is well defined.

3.3 LEMMA

Let the sequence $\{z^k := (u_k, s^k, t^k, x^k)\}$ be generated by Algorithm 3.1. Then

(i) $z^k \in \Omega$ for any $k \ge 0$.

(ii) $u^k > 0$ for all $k \ge 0$.

(iii) the sequence $\{\|H(z^k)\|\}$ is monotonically decreasing.

(iv) the sequence $\{u_k\}$ is monotonically decreasing.

Proof: (i) Since $z^0 \in \Omega$, we may assume without loss of generality that $z^k \in \Omega$. If the predictor step is accepted, then by (10), we have

$$u^k = \left(1/\beta\right)\left\|H(z^k)\right\|^{1+r}. \quad (21)$$

Combing (21) and (11), we can obtain $\hat{z}^k \in \Omega$; (22)

Otherwise, from $\hat{z}^k = z^k$ and the inductive assumption, we obtain that $\hat{z}^k \in \Omega$. By noting (12), we have

$$u^{k+1} = u^k + \theta^k \Delta u^k = \left(1-\theta^k\right)u^k + \left(\theta^k/\beta\right)\left\|H\left(\hat{z}^k\right)\right\|. \quad (23)$$

In addition, from (20) we know that there exist $\theta^k \in (0,1)$ such that

$$\left\|H\left(z^{k+1}\right)\right\| \le \left[1 - \sigma\left(1 - 1/\beta\right)\theta^k\right]\left\|H\left(\hat{z}^k\right)\right\| \le \left\|H\left(\hat{z}^k\right)\right\|. \quad (24)$$

Therefore, by (22)-(24), we can obtain that

$$\begin{aligned}
&\beta u^{k+1} - \left\|H\left(z^{k+1}\right)\right\| \\
&= \beta\left[\left(1-\theta^k\right)u^k + \left(\theta^k/\beta\right)\left\|H\left(\hat{z}^k\right)\right\|\right] - \left\|H\left(z^{k+1}\right)\right\| \\
&\ge \left(1-\theta^k\right)\left\|H\left(\hat{z}^k\right)\right\| + \theta^k\left\|H\left(\hat{z}^k\right)\right\| - \left\|H\left(z^{k+1}\right)\right\| \\
&\ge \left(1-\theta^k\right)\left\|H\left(\hat{z}^k\right)\right\| + \theta^k\left\|H\left(\hat{z}^k\right)\right\| - \left\|H\left(\hat{z}^k\right)\right\| = 0,
\end{aligned}$$

which indicates that $z^{k+1} \in \Omega$.

(ii) Since $u^0 > 0$, we may assume that $u^k > 0$ for any given $k \ge 0$. We know that $u^k > 0$; thus, it follows from (23) that $u^{k+1} > 0$. Hence, $u^k > 0$ for any $k \ge 0$.

(iii) If the predictor step (step 2) is not accepted at the k th iterate, then (24) implies that $\left\|H\left(z^{k+1}\right)\right\| \le \left\|H\left(\hat{z}^k\right)\right\| = \left\|H\left(z^k\right)\right\|$, and the desired result has been obtained; otherwise, (11) and $\left\|H\left(z^k\right)\right\| < 1$ imply that $\left\|H\left(\hat{z}^k\right)\right\| \le \left\|H\left(z^k\right)\right\|^{1+r} < \left\|H\left(z^k\right)\right\|$.

Hence, $\left\|H\left(z^{k+1}\right)\right\| \le \left\|H\left(z^k\right)\right\|$, for any $k \ge 0$.

(iv) For any $k \ge 0$, it follows from (22) and (23) that

$$u^{k+1} = \left(1-\theta^k\right)u^k + \left(\theta^k \big/ \beta\right)\left\|H\left(\hat{z}^k\right)\right\| \le u^k . \tag{25}$$

If the predictor step (step 2) is not accepted at the k th iterate, then (25) gives the desired result; otherwise, from (21) $\left\|H\left(z^k\right)\right\| < 1$ and $z^k \in \Omega$.

We have

$$u^k = \left(1 \big/ \beta\right)\left\|H\left(z^k\right)\right\|^{1+r} < u^k . \tag{26}$$

Combining (25) and (26), we obtain that $u^{k+1} \le u^k$ holds for any $k \ge 0$

4 Convergence Properties

In this section, we present the global convergence property of the Algorithm 3.1.

4.1 ASSUMPTION

The solution set of GNCP (1.1) is nonempty.

4.2 ASSUMPTION

For $f(x), g(x): R^n \to R^n$, there exists a constant $\rho > 0$ such that $\max_{i\in I}\left[f_i(x) - f_i(y)\right]\left[g_i(x) - g_i(y)\right] \ge \rho\|x-y\|^2$ for any all $x, y \in R^n$.

4.1 LEMMA

Let $\phi_p(u,a,b)$ be defined by Eq (3), and $\bar{u} > 0$. Assume that $\left\{\left(u_k, a^k, b^k\right)\right\}$ is a sequence such that

(i) $0 < u_k \le \bar{u}$ for $k \in I$.

(ii) either $a^k, b^k \to +\infty$ or $a^k \to -\infty$ or $b^k \to -\infty$ as $k \to +\infty$.

Then $\left|\phi_p\left(u_k, a^k, b^k\right)\right| \to +\infty$ as $k \to +\infty$.

Proof. By rationalizing the numerator, we have

$$\left|\phi_p\left(u_k, a^k, b^k\right) - \phi_p\left(a^k, b^k\right)\right| = \left|\sqrt[p]{\left|a^k\right|^p + \left|b^k\right|^p + u_k} - \sqrt[p]{\left|a^k\right|^p + \left|b^k\right|^p}\right|$$

$$= \frac{u_k}{\left(\sqrt[p]{\left|a^k\right|^p + \left|b^k\right|^p + u_k}\right)^{p-1} + \left(\sqrt[p]{\left|a^k\right|^p + \left|b^k\right|^p + u_k}\right)^{p-2}\sqrt[p]{\left|a^k\right|^p + \left|b^k\right|^p} + \cdots + \left(\sqrt[p]{\left|a^k\right|^p + \left|b^k\right|^p}\right)^{p-1}} .$$

$$\le \frac{u_k}{\left(\sqrt[p]{u_k}\right)^{p-1}} = \sqrt[p]{u_k} \le \sqrt[p]{\bar{u}}$$

This combining with Lemma 2.5 implies that the result of the Lemma 4.1holds.

4.2 LEMMA

Let $f, g: R^n \to R^n$ be a Lipschitz continuous functions satisfying Assumption 4.2 for some $\rho > 0$ and H be defined by equation (5).

Suppose that $\bar{u} > 0$. Then for any $u \in \left[0, \bar{u}\right]$, the function H is coercive, i.e., for any sequence $\left\{\left(u_k, s^k, t^k, x^k\right)\right\}$ which satisfies $\left\|z^k\right\| \to +\infty$ as $k \to +\infty$ and $u_k \in \left[0, \bar{u}\right]$ for all k, it follows that $\lim_{k\to+\infty}\left\|H\left(z^k\right)\right\| = +\infty$.

Proof: Suppose the result of the lemma is not true. Then there exists an unbounded sequence $\left\{z^k\right\}$ such that $u^k \in \left(0, \bar{u}\right)$ for all k and $\left\{H\left(z^k\right)\right\}$ is bounded. From (5), it follows that

$$\left\|H\left(z^k\right)\right\|^2 = u_k^2 + \left\|s^k - f\left(x^k\right)\right\|^2 + \left\|t^k - g\left(x^k\right)\right\|^2 + \left\|\Phi_p\left(z^k\right)\right\|^2 . \tag{27}$$

Since $\left\{\left\|H\left(z^k\right)\right\|\right\}$ is bounded, by (27), we have $\left\{s^k - f\left(x^k\right)\right\}$, $\left\{t^k - g\left(x^k\right)\right\}$ and $\left\{\Phi_p\left(z^k\right)\right\}$ are bounded. Since the sequence $\left\{z^k\right\}$ is unbounded, we have at least one of the sequences $\left\{x^k\right\}, \left\{s^k\right\}$, and $\left\{t^k\right\}$ is unbounded. We assume that $\left\{x^k\right\}$ is unbounded and thus a contr-adiction is obtained. From Assumption4.2, it follows that

$$\rho\|x^k\|^2 \le \max_{i\in I}\left[f_i(x^k)-f_i(0)\right]\left[g_i(x^k)-g_i(0)\right] \le \left[f_{i_0}(x^k)-f_{i_0}(0)\right]\left[g_{i_0}(x^k)-g_{i_0}(0)\right] \quad (28)$$

for some index $i_0 \in I$ independent of k and suitable subsequence of $\{x^k\}$. Form (28), we can obtain that either $\{f_{i_0}(x^k)\}$ or $\{g_{i_0}(x^k)\}$ is unbounded. We prove that both sequences are actually unbounded. Suppose that this is not true. Without loss of generality, we assume that $\{g_{i_0}(x^k)\}$ is unbounded and $\{f_{i_0}(x^k)\}$ is bounded. Then by (28), there exists a positive constant m such that

$$\rho\|x^k\|^2 \le m\left[g_{i_0}(x^k)-g_{i_0}(0)\right] \le Lm\|x^k\|^2, \quad (29)$$

where the second inequality follows from the fact g is Lipschitz continuous with the Lipschitz constant $L>0$. This contradicts the fact that $\{x^k\}$ is unbounded. Therefore, $\{f_{i_0}(x^k)\}$ and $\{g_{i_0}(x^k)\}$ are both unbounded for some index $i_0 \in I$, which implies that the sequences $\{s_{i_0}^k\}$ and $\{t_{i_0}^k\}$ are unbounded. Then by lemma 4.1, subsequence if necessary, we have $\left|\phi_p(u_k, s_{i_0}^k, t_{i_0}^k)\right| \to +\infty$ as $k \to +\infty$, which implies that $\{\Phi_p(u_k, s^k, t^k, x^k)\}$ is unbounded. This contradicts that $\{\Phi_p(z^k)\}$ is bounded. So $\{x^k\}$ is bounded. From the boundness of $\{x^k\}$, it follows that $\{s^k\}$ and $\{t^k\}$ are bounded. In fact, since f, g are continuous, it follows that $\{f(x^k)\}$ and $\{g(x^k)\}$ are bounded. Since $\{s^k - f(x^k)\}$ and $\{t^k - g(x^k)\}$ are bounded, the sequences $\{s^k\}$ and $\{t^k\}$ are bounded. Therefore, the sequence $\{z^k\}$ is bounded. A contradiction is derived. The proof is completed.

4.3 LEMMA

Let $f, g: R^n \to R^n$ be Lipschitz continuous functions satisfying Assumption 4.2 for some $\rho > 0$, and f, g are continuously differentiable. Let sequence $\{z^k\}$ be generated by algorithm 3.1. Then $\lim_{k\to+\infty}\|H(z^k)\| = 0$ and $\lim_{k\to+\infty} u_k = 0$.

Proof: The proof is similar to that of Lemma in [22].

By using Lemma3.2 (iii) and $z^k \in \Omega$, we have $0 \le \|H(z^{k+1})\| \le \|H(z^k)\| \le \beta u_k \le \beta u_0$.

Thus there exists a scalar $h \ge 0$ such that $\lim_{k\to\infty}\|H(z^k)\| = h$. If $h = 0$, then the desired result has been obtained. In the following, we assume that $h > 0$. From Lemma3.2 (iii) and $z^k \in \Omega$, we have $0 < \left(1/\beta\right) h \le \left(1/\beta\right)\|H(z^{k+1})\| \le u_{k+1} \le u_k \le u_0$.

Thus, by using Lemma 4.2, we obtain that the sequence $\{z^k\}$ is bounded. Subsequence if necessary, we may assume that there exists a point $z^* = (u_*, s^*, t^*, x^*)$ such that $\lim_{k\to\infty} z^k = z^*$ and $\lim_{k\to\infty}\|H(z^k)\| = \|H(z^*)\|$. It is easy to see that $\|H(z^*)\| = h$. Hence, $u_* > 0$, since $\|H(z^*)\| > 0$. From $\|H(z^*)\| > 0$, we have $\lim_{k\to\infty}\theta_k = 0$, thus, the step size $\bar{\lambda}_k := \lambda_k/\delta$ does not satisfy the line search criterion (13) for any sufficiently large k; i.e., the following inequality holds: $\left\|H\left(\hat{z}^k + \bar{\theta}^k \Delta\tilde{z}^k\right)\right\| > \left[1-\sigma\left(1-1/\beta\right)\bar{\theta}^k\right]\left\|H\left(\hat{z}^k\right)\right\|$, for any sufficiently large k, which implies that $\left(\left\|H\left(\hat{z}^k + \bar{\theta}^k\Delta\tilde{z}^k\right)\right\| - \left\|H\left(\hat{z}^k\right)\right\|\right)/\bar{\theta}^k > -\sigma\left(1-1/\beta\right)\left\|H\left(\hat{z}^k\right)\right\|$.

From $u^* > 0$, we know that $H(z)$ is continuously differentiable at z^*. Let $k \to \infty$, then the above inequality gives

$$\frac{H(z^*)^{\mathrm{T}}}{\|H(z^*)\|} H'(z^*)\Delta z^* \ge -\sigma\left(1-1/\beta\right)\|H(z^*)\|. \quad (30)$$

In addition, by taking the limit on (30) we have

$$H'(z^*)\Delta z^* = -H(z^*) + \left(1/\beta\right)\|H(z^*)\| e^0. \quad (31)$$

Combing (30) and (31) we have $-1 + 1/\beta + \sigma\left(1-1/\beta\right) \ge 0$ which contradicts the fact that $\beta > 1$ and $\sigma \in (0,1)$. This implies that $H(z^*) = 0$. Thus $u_* = 0$ by the definition of $H(z)$ and $H(z^*) = 0$.

4.4 LEMMA

Suppose that $f, g: R^n \to R^n$ be Lipschitz-continuous functions satisfying Assumption 4.2 for some $\rho > 0$, and f, g are continuously differentiable. Let sequence $\{z^k\}$ be generated by algorithm 3.1. Then sequence $\{z^k\}$ is bounded.

Proof: by Lemma 3.2 (iv), $\{u_k\}$ is monotonically decreasing and $u_k > 0$ for all $k \in I$, which implies that $0 < u_k \leq u_0$ for $k \in I$. In addition, by the definition of Ω, we can get $\left\|H\left(z^k\right)\right\| \leq \beta u_k$ for all $k \in I$. Thus, $\left\{H\left(z^k\right)\right\}$ is bounded. Then by Lemma 4.2, we can easily prove $\left\{z^k\right\}$ is bounded.

4.5 THEOREM

Suppose that $f, g : R^n \to R^n$ be Lipschitz continuous functions satisfying Assumption 4.2 for some $\rho > 0$, and f and g are continuously differentiable. Suppose that Assumption 4.2 is satisfied. Then the sequence $\left\{z^k\right\}$ generated by Algorithm 3.1 is bounded and any accumulation point of $\left\{x^k\right\}$ is a solution of GNCP (1).

Proof: By Lemma 4.4 we know that the sequence $\left\{z^k\right\}$ generated by Algorithm 3.1. is bounded. Without loss of generality, we write the convergent subsequence of $\left\{z^k\right\}$ as $\left\{z^k\right\}$. Let $z^* := \left(u_*, s^*, t^*, x^*\right)$ be the limit point of $\left\{z^k\right\}$. Then by Lemma 4.3, $u_* = 0$ and $H\left(z^*\right) = 0$. Thus, by a simple continuity discussion we prove that x^* is a solution of GNCP(1).

5 Conclusions

In this paper, a predictor-corrector smoothing Newton method is presented for the generalized nonlinear complementarity problem based on a class of smoothing functions. Some good properties are given by the proposed a class of smoothing functions. Under suitable conditions, the boundedness of the iteration sequence generated by the proposed smoothing Newton method is proved and any accumulation point of the generated sequence is a solution of the generalized nonlinear complementarity problem the global convergence of the proposed algorithm is established.

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