# Theoretical analyses and numerical simulation of the interaction time and the separation time of two elastic bars after the loading of a triangular wave

**Wei Zhong*, Ming Lei, Zhou Tian**

*Northwest Institute of Nuclear Technology, Xi'an 710024, Shaanxi Province, China*

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**Abstract**

With theoretical analyses, the stress-wave propagation and reflection between two elastic bars, whose left side achieved a triangular wave load, is mainly studied. Considering the effects of pulse width of the load, length and wave impedance of the bars and the relationship between the length of the load wave and the two bars, the regularities of the stress-wave propagation and reflection between the two bars is analysed, and the formulas for calculating the interaction time and the separation time of the two bars are derived, in three different conditions. The same problems in three different conditions are simulated by using AUTODYN, and curves of displacements of points at contact surface of the two bars varying the time are given. According these curves, the simulating results of the separation time can be obtained. By the comparison of the simulating results and the theoretical calculated results using formulas derived in this paper, the correctness of the theoretical analyses and formulas here is demonstrated.

*Keywords:* triangular wave load, elastic bar, separation time, theoretical analyses, AUTODYN

## 1 Introduction

With the rapid development of high-tech, the field of such relate to explosion and shock problem as national defence technology, aerospace industry, new material technology, etc, is more and more widely, especially in areas such as the arms penetration and explosion, high-speed impact, dynamic response materials, structural damage and other protective aspects [1]. As we know, the study of those problem mentioned above is related to the stress wave propagation. Stress wave propagation characteristics in multi-layer composite structure are a newly rising research field [2-4]. Although the current experimental study of homogeneous material for macroscopic dynamic mechanical behaviour and impulse response has developed many ripe methods, such as Taylor test, split Hopkinson pressure bar, flat impact and detonation technology [5-7], the structures of the actual project are usually layer structures stacked by a variety of materials with different physical properties. Because of light quality, strong design, good performance and strong impact absorbing characteristics of these kind of structures, they have become one of the hot issues of structural engineering applications in recent years [8]. Taking into account that the multi-layer dielectric stress wave propagation problem in the actual project is very complicated, as a theoretical analysis here, the research was focused on the stress wave propagation and reflection of two elastic bars after the loading of a triangular wave. Recently, the most widely application of experimental studies of such problem is the split Hopkinson pressure bar's (SHPB) [9, 10], also known as the Kolsky bar. SHPB technology is on the basis of one-dimensional stress wave theory, which requires the long-thin-bar is linear, isotropic and the dispersion effect can be ignored, meanwhile, the cross-section of the bar in the axial direction is assumed to be a constant and the bar to maintain flexibility and uniform stress state in the process of loading and unloading [11]. As a theoretical analysis of the situation above, the theoretical analysis of this article and the following derivation is also based on the same requirements above.

## 2 Theoretical analysis and calculation formula derivation

In Figure 1, two elastic bars contact with each other, and a input triangular wave loads with the wavelength of $\lambda$ acts on the left of bar 1. In the following, the interaction time and the separation time of the two bars will be analyzed. $L_1$ and $L_2$ is respectively the length of bar 1 and bar 2, as well as $c_1$ and $c_2$ the triangular-wave propagation velocity in bar 1 and bar 2. We assume that the two bars are the same, that is to say, $c_1$ and $c_2$ are equal, and can be taken as $c$. Next, we will prove that the interaction time and the separation time of the two bars are determined by $\lambda$, $L_2$, $c_2$, and the relative relationship between $\lambda$ and $L_2$.

****Corresponding author*** e-mail: lengshui222@163.com

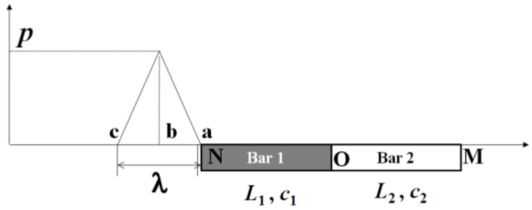


FIGURE 1 Model sketch of this problem.

## 2.1 CASE 1: $\lambda/2<L_2$

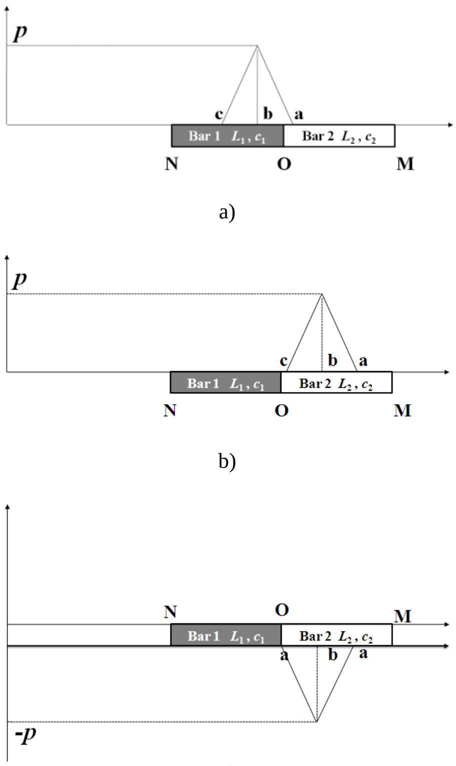


FIGURE 2 Diagram of the stress-wave propagation when $\lambda/2 < L_2$

Since the load is linear and the bars are elastic, the relationship between stress in bars and the input load is linear. Therefore, for easy discussion, the stress of bars will be expressed in the input load in following analysis.

When the front of the stress wave come location O, which is the contact surface of bar 1 and bar 2, the interaction force between the two bars occurs, and compressive stress appears in these two bars, as shown in Figure 2a. When the end of the stress wave passes location O, the interaction force disappears, and the two bars have a same velocity, as shown in Figure 2b. Then, the interaction time of bar 1 and bar 2 can be obtained by:

$$t_{\text{interaction}} = \frac{\lambda}{c}. \quad (1)$$

When the stress wave comes to location M, which is the right end of bar 2, stress wave reflection occurs at the free surface, and an extension wave propagation to the left appears [12]. Particles of the bar will get a velocity to the right after the pass of the extension wave. So when the extension wave comes to the location O, bar 1 and bar 2 will separate from each other, as shown in Figure 2(c). Then, the separation time of bar 1 and bar 2 can be obtained by:

$$t_{\text{separation}} = \frac{L_1 + 2L_2}{c}. \quad (2)$$

## 2.2 CASE 2: $\lambda/2 > L_2$ and $\lambda/2 < (L_1+L_2)$

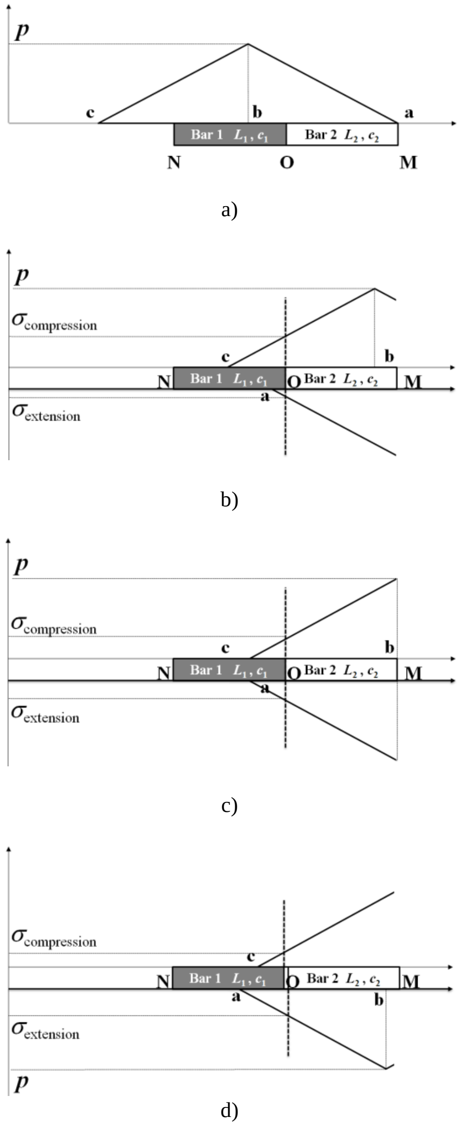


FIGURE 3 Diagram of the stress-wave propagation when $\lambda/2 > L2$ and $\lambda/2 < (L1+L2)$

When the front of the stress wave come location O, the interaction force between the two bars occurs, and compressive stress appears in these two bars, as shown in Figure 3a. When the front of the stress wave comes to the right end of bar 2 but the mid of the stress wave has not come to location O, stress wave reflection occurs at the free surface, and an extension wave propagation to the left is formed. At the time of this extension wave comes to the location O, the end of the stress wave has not come to the location O yet, and meanwhile, the compressive stress is higher than the stress generated by the reflection extension wave, so the interaction force between bar 1 and bar 2 is always maintained, which will keep the two bars contacting with each other and moving together, as shown in Figure 3b. As the mid of the stress wave comes to the right end of bar 2 and the front of the tensile stress wave passes the location O, although the end of the stress wave has not come to the location O, the tensile stress at location O is equal to the input stress, which means the stress at location O is zero at this time, which can be seen in Figure 3c. As shown in Figure 3d, with the continuing wave propagation, bar 1 and bar 2 will separate from each other, and the separation time of bar 1 and bar 2 can be obtained by:

$$t_{\text{separation}} = \frac{0.5\lambda + L_1 + L_2}{c}\,. \tag{3}$$

According to the analysis above, it is easy to get the interaction time of bar 1 and bar 2 by:

$$t_{\text{interaction}} = \frac{0.5\lambda + L_2}{c}\,. \tag{4}$$

## 2.3 CASE 3: $\lambda/2 > (L_1+L_2)$

This is a very complex case, and the interaction time and separation time are closely related to the waveform of the input stress. In order to make the research more convenient, a special stress wave propagation graph was made, which was shown in Figure 4.

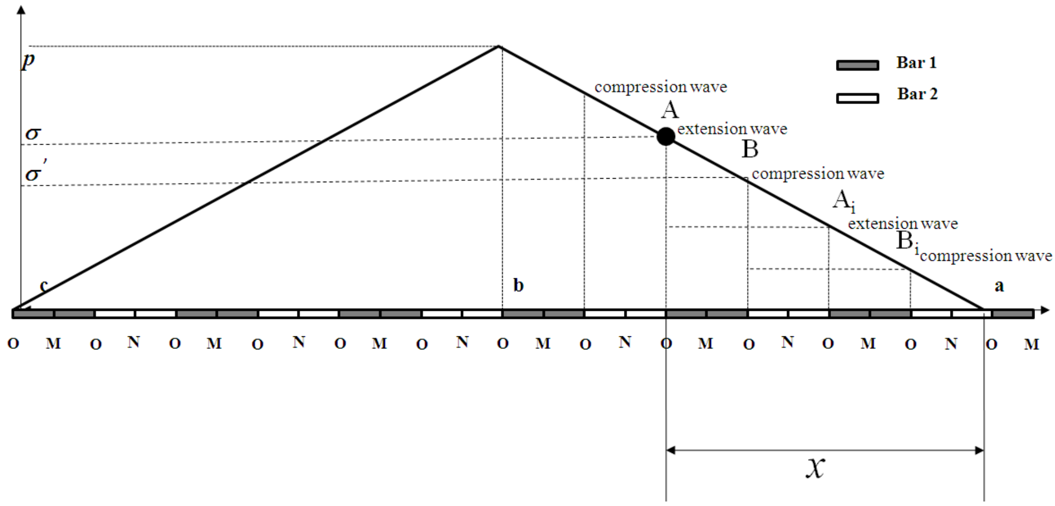


FIGURE 4 Diagram of the stress-wave propagation when $\lambda/2 > (L_1+L_2)$

Assume the distance between the front wave and the extension wave A is $x$ at the time of the extension wave A comes to location O the first time. The relationship between the tensile stress and the compression stress were given in Figure 4. For example, the stress in location B appears, when the extension wave comes to the M free surface - that is, after the input stress wave reflected in the left free surface of bar 1, a reflect compression wave formed, with the wave propagation continuing a length of bar 1, the compressive stress in contact surface of bar 1 and bar 2 appears. Other cases have the similar analysis above.

It is easy to be proved that there is a range of $x$, and the range is not larger than half of the wavelength. If $x$ is equal to half of the wavelength, the location A will be the top of the input wave when the reflection extension wave comes to location A at the first time. And this location A is where the tensile stress of the contact surface could rise to a maximum. The Equation of the tensile stress and the compressive stress will be derived next.

### *2.3.1 Tensile stress*

The slope of the triangular wave is:

$$k = \frac{p}{\lambda/2}\,. \tag{5}$$

From geometrical relations in Figure 4, the slope on the expression of stress and $x$ is:

$$k = \frac{\sigma}{x}\,.$$

Then:

$$\sigma = k \cdot x\,. \tag{6}$$

According to Figure 4, it is easy to know the number of the tensile stress waves reflected back and forth at this point is:

$$N = \left[\frac{x}{2(L_1+L_2)}\right] + 13 \tag{7}$$

The symbol【 】indicates that the number is rounded to the nearest whole number.

For the *i*-th reflection ($i \geq 2$), the tensile stress satisfies the following Equation:

$$k = \frac{\sigma - \sigma_i}{(i-1)\times[2(L_1+L_2)]} .$$

Then:

$$\sigma_i = \sigma - k\times(i-1)\times[2(L_1+L_2)] . \tag{8}$$

All the tensile stress at the contact surface is the sum of $\sigma_i$:

$$\sigma_{\text{tensile}} = \sigma + \sum_{i=2}^{N}\{\sigma - k\times(i-1)\times[2(L_1+L_2)]\} . \tag{9}$$

*2.3.2 Compressive stress*

The compressive stress at location B can be obtained from geometrical relations in Figure 4:

$$\frac{\sigma\text{-}\sigma'}{2L_1} = k .$$

Then:

$$\sigma' = \sigma - k\times 2L_1 \tag{10}$$

As the same, according to Figure 4, it is easy to know the number of the tensile stress waves reflected back and forth at this point is:

$$M = 【\frac{x-2L_1}{2(L_1+L_2)}】+1 . \tag{11}$$

For the *i*-th reflection ($i \geq 2$), the compressive stress satisfies the following equation:

$$k = \frac{\sigma' - \sigma'_i}{(i-1)\times[2(L_1+L_2)]} .$$

Then:

$$\sigma'_i = \sigma' - k\times(i-1)\times[2(L_1+L_2)] \tag{12}$$

So, all the tensile stress at the contact surface is the sum of $\sigma'_i$:

$$\sigma_{\text{compressive\_reflect}} = \sigma' + \sum_{i=2}^{M}\{\sigma' - k\times(i-1)\times[2(L_1+L_2)]\} \tag{13}$$

When $(\frac{\lambda}{2} - x) \geq 2L_2$, we get:

$$\frac{\sigma_{\text{compressive\_incident}} - \sigma}{2L_2} = k .$$

Then:

$$\sigma_{\text{compressive\_incident}} = \sigma + k\times 2L_2 . \tag{14-1}$$

When $0 < (\frac{\lambda}{2} - x) < 2L_2$, we get:

$$\begin{cases} \dfrac{p-\sigma}{\delta} = k \\ \dfrac{p-\sigma_{\text{compressive\_incident}}}{2L_2-\delta} = k \end{cases} .$$

Where $\delta$ is the distance between location A and the mid of the wave, so:

$$\sigma_{\text{compressive\_incident}} = 2p - 2L_2k - \delta \tag{14-2}$$

From (13), (14-1) and (14-2), we obtained the total compressive stress:

$$\sigma_{\text{compressive}} = \sigma_{\text{compressive\_reflection}} + \sigma_{\text{compressive\_incident}}$$

$$= \sigma' + \sum_{i=2}^{M}\{\sigma' - k\times(i-1)\times[2(L_1+L_2)]\} \tag{15}$$

$$\begin{cases} +\sigma + k\times 2L_2 & (\frac{\lambda}{2}-x) \geq 2L_2 \\ +2p - 2L_2k - \sigma & 0 < (\frac{\lambda}{2}-x) < 2L_2 \end{cases}$$

If tensile stress is larger than compressive stress on the contact surface, bar 2 and bar 1 will separate from each other.

Assume $\sigma_{\text{tensile}} = \sigma_{\text{compressive}}$, we get $x_{\text{separation}}$:

$$x_{\text{separation}} = \begin{cases} \dfrac{1}{[N-(M+1)]}\times\{2L_2 - 2L_1M + \sum\limits_{i=M+1}^{N}[k(i-1)\times 2(L_1+L_2)]\} & (\frac{\lambda}{2}-x) \geq 2L_2 \\ \dfrac{1}{[N-(M-1)]}\times\{2p/k - 2L_2 - 2L_1M + \sum\limits_{i=M+1}^{N}[k(i-1)\times 2(L_1+L_2)]\} & 0 < (\frac{\lambda}{2}-x) < 2L_2 \end{cases} \tag{16}$$

and:

$$\begin{cases} k = \dfrac{p}{\lambda/2} \\ N = 【\dfrac{x}{2(L_1+L_2)}】+1 \\ M = 【\dfrac{x-2L_1}{2(L_1+L_2)}】+1 \end{cases} . \tag{17}$$

So the separation time of bar 2 and bar 1 can be got by:

$$t_{\text{separation}} = \frac{x_{\text{separation}} + L_1 + 2L_2}{c} . \tag{18}$$

And the interaction time:

$$t_{\text{interaction}} = \frac{x_{\text{separation}} + 2L_2}{c} . \tag{19}$$

Equations (16) and (18) shows that, the separation time and the interaction time of bar 1 and bar 2 are determined by the length of bar 1 and bar 2, the length of the stress wave and the wave resistance (equals to the product of stress wave propagation velocity and the density of the bars).

If the calculation results of Equation (16) is that $x_{\text{separation}}$ is larger than length of the wave, bar 1 and bar 2 will not separate, although the reflective tensile stress of the end of the input wave has arrived at the contact surface of the two bars. Then, for this situation, the two bars will always move together, and never separate from each other. This situation is actually a completely inelastic collision.

## 3 Numerical simulation and verification

In order to verify the correctness of the theoretical analysis above, numerical simulation of the three cases above has been carried out by AUTODYN, and results of the separation time were given. The comparison between these numerical results and the analytical solutions according to the theoretical analysis in this paper has been done also.

In the following simulation and computation, bar 1 and bar 2 are both the 20 # steel ($E$=210 GPa, $\rho$=7.82 g/cm$^3$, $\mu$=0.28), as well as the length both 1 metre.

### 3.1 CASE 1: $\lambda/2<L_2$

Conditions of the input triangle wave: $\sigma_0$=300 Mpa, $\lambda$=292.96 mm.

The numerical simulation has been done by AUTODYN, and the results of displacement of points at the right side of bar 1 and the left side of bar 2 varying the time was shown in Figure 5.

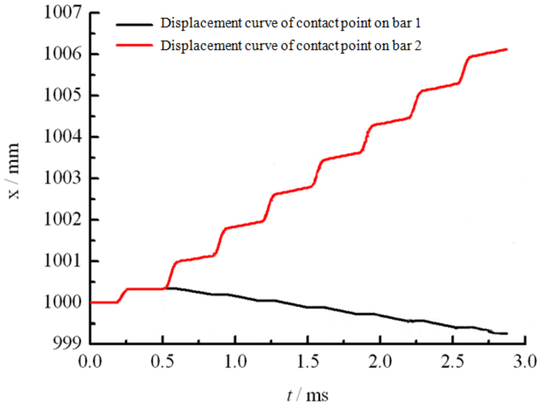


FIGURE 5 Curves of displacement of points at the right side of Bar 1 and the left side of Bar 2 varying the time

According to the simulations and Figure 5, we got the separation time of bar 1 and bar 2 was about 0.5117 ms.

As the elastic wave velocity in the bars is:

$$c = \sqrt{\frac{E(1-\mu)}{(1+\mu)(1-2\mu)\rho}} = 5859.24 \quad \text{mm / ms} .$$

According to Equation (2), we got:

$$t_{\text{separation}} = \frac{L_1 + 2L_2}{c} = 0.5120 \quad \text{ms} .$$

The relative error was:

$$\eta = \frac{0.5120 - 0.5117}{0.5117} = 0.058\% .$$

And the interaction time of bar 1 and bar 2 could also been obtained from Equation (1):

$$t_{\text{interaction}} = \frac{\lambda}{c} = 0.1 \quad \text{ms} .$$

### 3.2 CASE 2: $\lambda/2 > L_2$ and $\lambda/2 < (L_1+L_2)$.

Conditions of the input triangle wave: $\sigma_0$=300 Mpa, $\lambda$=1464.80 mm.

The numerical simulation has been done by AUTODYN, and the results of displacement of points at the right side of bar 1 and the left side of bar 2 varying the time was shown in Figure 6.

According to the simulations and Figure 6, we got the separation time of bar 1 and bar 2 was about 0.63 ms.

As the elastic wave velocity in the bars is:

$$c = \sqrt{\frac{E(1-\mu)}{(1+\mu)(1-2\mu)\rho}} = 5859.24 \quad \text{mm/ms} .$$

According to Equation (2), we got:

$$t_{\text{separation}} = \frac{0.5\lambda + L_1 + L_2}{c} = 0.5913 \quad \text{ms} .$$

The relative error was:

$$\eta = \frac{0.63 - 0.5913}{0.63} = 6.1\% \cdot$$

The interaction time of bar 1 and bar 2 could also been obtained from Equation (1):

$$t_{\text{interaction}} = \frac{0.5\lambda + L_2}{c} = 0.4206 \quad \text{ms} .$$

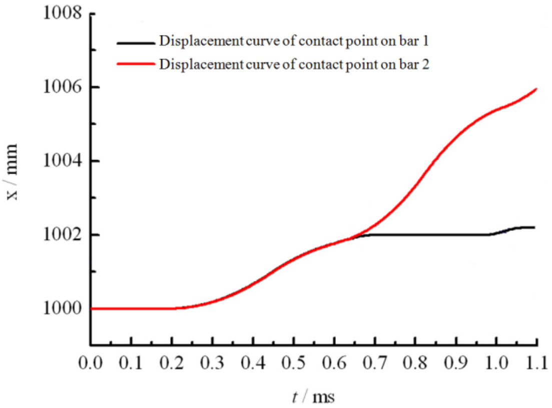


FIGURE 6 Curves of displacement of points at the right side of Bar 1 and the left side of Bar 2 varying the time

3.3 CASE 3: $\lambda/2 > (L_1+L_2)$

Conditions of the input triangle wave: $\sigma_0$=300 Mpa, $\lambda$=14648 mm.

The numerical simulation has been done by AUTODYN, and the results of displacement and velocity of points at the right side of bar 1 and the left side of bar 2 varying the time was shown in Figure 7 and Figure 8.

With reference to Figure 7 and Figure 8 we could see that bar 1 and bar 2 almost always move together, and don't separate from each other. Figure 8 showed that, during the first 5 ms, velocities of bar 1 and bar 2 have been increasing. After the first 5 ms, both bars travelled in a constant velocity.

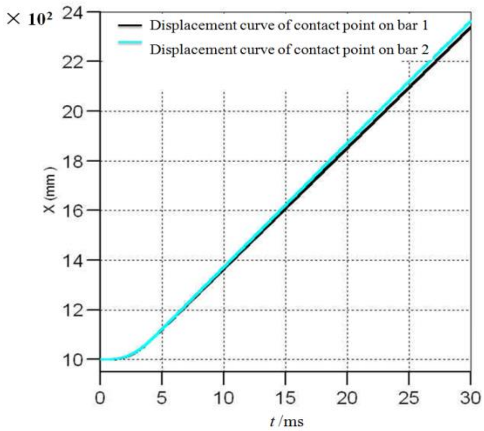


FIGURE 7 Curves of displacement of points at the right side of Bar 1 and the left side of Bar 2 varying the time

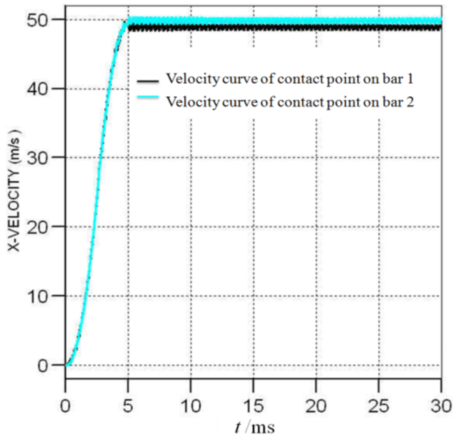


FIGURE 8 Curves of velocity of points at the right side of Bar 1 and the left side of Bar 2 varying the time

As the elastic wave velocity in the bars is:

$$c = \sqrt{\frac{E(1-\mu)}{(1+\mu)(1-2\mu)\rho}} = 5859.24 \quad \text{mm/ms}.$$

According to the analysis in section II, we could get the following result by calculation:

$$x_{separation} > \lambda . \tag{20}$$

Solving the Equation (16) directly is a difficult work, but Equation (20) can be verified indirectly: substituting any $x_{separation}$ which was smaller than the wavelength $\lambda$ " in Equation (9) and Equation (15), and solving these equations, we could obtain that $\sigma_{compressive}$ was always larger than $\sigma_{tensile}$. This result means that the compressive stress is always larger than the tensile stress at the contact surface of bar 1 and bar2. So the two bars didn't separate from each other. And this conclusion is consistent with the numerical simulation.

## 4 Conclusions

On the basis of the basic theory and laws of elastic wave propagation and reflection, stress wave propagation and reflection between two contact elastic bars, one of which was loaded by a triangle wave on the left, has been researched. For three different cases, formulas of separation time and interaction time have been derived. Numerical simulation of these three cases has also been done by using AUTODYN, and comparison of the numerical simulation and the results obtained by the method of this paper showed that the formulas derived here is correct.

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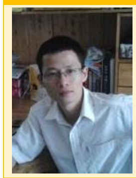
## Authors



**Wei Zhong, China**

**Current position, grades**: research assistant in Northwest Institute of Nuclear Technology, China.
**University study**: Master's degree in mechanical engineering, Northwest Institute of Nuclear Technology, Xi'an, China, 2012.
**Research activities**: mathematics, fluid dynamics, mechanical engineering, explosion mechanics, chemical reaction kinetics, numerical simulation methods.

**Ming Lei, China**

**Current position, grades**: research assistant in Northwest Institute of Nuclear Technology, China.
**University study**: Master's degree in mechanical engineering, Northwest Institute of Nuclear Technology, Xi'an, China, 2004.
**Research activities**: mathematics, finite element method, mechanical engineering, explosion mechanics, numerical simulation methods.

**Zhou Tian, China**

**Current position, grades**: professor in Northwest Institute of Nuclear Technology, China.
**University study**: PhD degree in nuclear science and technology, Northwest Institute of Nuclear Technology, Xi'an, China, 2003.
**Research activities**: mathematics, physics, mechanical engineering, detonation and shock waves, explosion mechanics, numerical simulation methods.