Two duality problems for a class of multi-objective fractional programming

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Received 1 March 2014, www.cmnt.lv

Abstract

In this article, we investigate the duality results for a class of non-differentiable multi-objective fractional programming problems. The parametric dual models and Wolfe dual models are formulated for this fractional programming. Weak, strong and strict converse duality theorems are established and proved based on the generalized invexity assumptions. Some previous duality results for differentiable multi-objective programming problems turn out to be special cases for the results described in the paper.

Keywords: invexity, multi-objective fractional programming, parametric dual, Wolfe dual

1 Introduction

The term multi-objective programming is an extension of mathematical programming where a scalar valued objective function is replaced by a vector function. Many approaches for multi-objective programming problems have been explored in considerable details, see for example [1,2]. The multi-objective fractional programming refers to a multi-objective problem where the objective functions are quotients of two functions. Recently, the studies about the optimizations of the multi-objective fractional programming problems have also been a focal issue due to in many practical optimization problems the objective functions are quotients. For example, in [3-5] some necessary and sufficient optimality conditions for a feasible point to be an efficient solution for non-differentiable multi-objective fractional programming problems were obtained in the framework of generalized convexity.

Furthermore, duality plays a fundamental role in mathematics, especially in optimization. It has not only used in many theoretical and computational developments in mathematical programming itself but also used in economics, control theory, business problems and other diverse fields. It is not surprising that duality is one of the important topics in multi-objective optimization. A large literature was developed around the duality in multi-objective fractional optimization under the generalized convexity assumption. For example, the results in [6-11] have weakened the convexity hypothesis and made the important contribution in duality theorems.

In this paper, motivated by the above work, the duality results are obtained for a class of multi-objective fractional programming problem under the assumptions of $(b,\alpha)-\rho-(\eta,\theta)$ – invexity.

2 Definitions and preliminaries

Definition 1. Let $X \subseteq R^n$. The function $f: X \to R$ is locally Lipschitz on X, if there exists a positive constant k, such that:

$$\left| f(x_1) - f(x_2) \right| \le k \left\| x_1 - x_2 \right\|,\ \forall x_1, x_2 \in X . \tag{1}$$

Definition 2. If $f: X \to R$ is locally Lipschitz on X, the generalized Clarke directional derivative of f at $x \in X$ in the direction $d \in R^n$ is defined by:

$$f^0(x;d) = \lim_{(y,t)\to(x,0^+)} \sup \frac{f(y+td)-f(y)}{t} . \tag{2}$$

The generalized sub-gradient of a locally Lipschitz function f at $x \in X$ is defined by:

$$\partial f(x) = \{\xi \in R^n : f^0(x;d) \ge \langle \xi, d \rangle, \forall d \in R^n\} . \tag{3}$$

Throughout this paper, we will use the following the definitions given in [11], we always assume that X is an open subset of R^n, $b: X \times X \times [0,1] \to R^+$, $b(x_1,x_2) = \lim_{\lambda\to 0^+} b(x_1,x_2,\lambda) \ge 0, \alpha: X \times X \to R^+ \setminus \{0\}, \eta: X \times X \to R^n$, $\theta: X \times X \to R^n, \rho \in R$.

Definition 3. A Lipschitz function $f: X \to R$ is said to be $(b,\alpha)-\rho-(\eta,\theta)$ – invex at $u \in X$, if there exists b,α,η,θ and ρ, such that:

$$b(x,u)[f(x)-f(u)] \ge \langle \alpha(x,u)\xi, \eta(x,u) \rangle + \rho \left\| \theta(x,u) \right\|^2 . \tag{4}$$

Remark 1. If in the above definition, we have strict inequality for any $x \ne u$, then we say that f is $(b,\alpha)-\rho-(\eta,\theta)$ – strictly invex at $u \in X$.

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Definition 4. A Lipschitz function $f: X \to R$ is said to be $(b,\alpha)-\rho-(\eta,\theta)$ – pseudo-invex at $u \in X$, if there exists b,α,η,θ and ρ, such that:

$$\langle \alpha(x,u)\xi, \eta(x,u)\rangle + \rho\|\theta(x,u)\|^2 \geq 0 \Rightarrow \\ b(x,u)[f(x)-f(u)] \geq 0, \forall x \in X, \xi \in \partial f(u), \tag{5}$$

Definition 5. A Lipschitz function $f: X \to R$ is said to be $(b,\alpha)-\rho-(\eta,\theta)$ – quasi-invex at $u \in X$, if there exists b,α,η,θ and ρ, such that:

$$b(x,u)[f(x)-f(u)] \leq 0 \Rightarrow \\ \langle \alpha(x,u)\xi, \eta(x,u)\rangle + \rho\|\theta(x,u)\|^2 \leq 0, \forall x \in X, \xi \in \partial f(u). \tag{6}$$

Definition 6. A Lipschitz function $f: X \to R$ is said to be $(b,\alpha)-\rho-(\eta,\theta)$ – strictly quasi-invex at $u \in X$, if there exists b,α,η,θ and ρ, such that:

$$b(x,u)[f(x)-f(u)] \leq 0 \Rightarrow \langle \alpha(x,u)\xi, \eta(x,u)\rangle + \\ \rho\|\theta(x,u)\|^2 < 0, \forall x \neq u \in X, \xi \in \partial f(u). \tag{7}$$

3 Parametric duality

In this section, we consider the parametric dual model for (MFP). The dual can be formulated as follows:

$$\text{(MFD}_1\text{)} \quad \begin{aligned} &\max v = (v_1, v_2, \cdots, v_p) \\ &s.t.\ 0 \in \sum_{i=1}^{p} \lambda_i[\partial f_i(u) - v_i \partial g_i(u)] + \sum_{j=1}^{m} \mu_j \partial h_j(u), \\ &f_i(u) - v_i g_i(u) \geqq 0, \text{for all } i = 1,2,\cdots,p, \\ &\sum_{j=1}^{m} \mu_j h_j(u) \geqq 0, \\ &\lambda \in \Lambda^{++}, \mu \in R_+^m, v \in R_+^p. \end{aligned}$$

Let:

$$D^0 = \{(u,\lambda,\mu,v) \in X \times \Lambda^{++} \times R_+^m \times R_+^p, | \\ 0 \in \sum_{i=1}^{p} \lambda_i[\partial f_i(u) - v_i \partial g_i(u)] + \sum_{j=1}^{m} \mu_j \partial h_j(u), f_i(u) - v_i g_i(u) \geqq \\ 0 \text{ with } i = 1,2,\cdots,p, \sum_{j=1}^{m} \mu_j h_j(u) \geqq 0\}$$

denote the feasible set of (MFD$_1$).

Theorem 1. (Weak Duality) Let $x^0 \in X^0$ and $(\bar{u},\bar{\lambda},\bar{\mu},\bar{v}) \in D^0$. Suppose that:

(i) f_i is $(b_i,\alpha)-\rho_i-(\eta,\theta)$ – invex at $\bar{u}$, $-g_i$ is $(b_i,\alpha)-\bar{\rho}_i-(\eta,\theta)$ – invex and regular at $\bar{u}$, $i=1,2,\cdots,p$;

(ii) h_j is $(c,\alpha)-\gamma_j-(\eta,\theta)$ – invex at $\bar{u}$, $j=1,2,\cdots,m$;

(iii) $b_i(\bar{x},\bar{u}) > 0, i=1,2,\cdots,p, c(\bar{x},\bar{u}) \geqq 0$;

(iv) $\sum_{i=1}^{p} \bar{\lambda}_i(\rho_i + \bar{v}_i\bar{\rho}_i) + \sum_{j=1}^{m} \bar{\mu}_j\gamma_j \geqq 0$.

Then $F(\bar{x}) \not\leq \bar{v}$.

Proof: Suppose contrary to the result that $F(\bar{x}) \leq \bar{v}$. That implies:

$$\frac{f_i(\bar{x})}{g_i(\bar{x})} \leqq \bar{v}_i, \text{for } i = 1,2,\cdots,p, \tag{8}$$

and:

$$\frac{f_k(\bar{x})}{g_k(\bar{x})} < \bar{v}_k, \text{for some } k \in \{1,2,\cdots,p\}. \tag{9}$$

That is:

$$f_i(\bar{x}) - \bar{v}_i g_i(\bar{x}) \leqq f_i(\bar{u}) - \bar{v}_i g_i(\bar{u}), \text{for } i = 1,2,\cdots,p, \tag{10}$$

and:

$$f_k(\bar{x}) - \bar{v}_k g_k(\bar{x}) < f_k(\bar{u}) - \bar{v}_k g_k(\bar{u}), \\ \text{for some } k \in \{1,2,\cdots,p\}. \tag{11}$$

Since $(\bar{u},\bar{\lambda},\bar{\mu},\bar{v}) \in D^0$ and $b_i(\bar{x},\bar{u}) > 0$, $i=1,2,\cdots,p$, the above inequalities yield:

$$\sum_{i=1}^{p} b_i(\bar{x},\bar{u})\bar{\lambda}_i[f_i(\bar{x}) - \bar{v}_i g_i(\bar{x})] < \\ \sum_{i=1}^{p} b_i(\bar{x},\bar{u})\bar{\lambda}_i[f_i(\bar{u}) - \bar{v}_i g_i(\bar{u})]. \tag{12}$$

Also, using $\bar{x} \in X^0$ and $c(\bar{x},\bar{u}) > 0$, we have:

$$c(\bar{x},\bar{u})\sum_{j=1}^{m} \bar{\mu}_j h_j(\bar{x}) \leqq c(\bar{x},\bar{u})\sum_{j=1}^{m} \bar{\mu}_j h_j(\bar{u}). \tag{13}$$

Adding the above two inequalities together, we get:

$$\sum_{i=1}^{p} b_i(\bar{x},\bar{u})\bar{\lambda}_i(f_i(\bar{x}) - \bar{v}_i g_i(\bar{x})) + c(\bar{x},\bar{u})\sum_{j=1}^{m} \bar{\mu}_j h_j(\bar{x}) < \\ \sum_{i=1}^{p} b_i(\bar{x},\bar{u})\bar{\lambda}_i(f_i(\bar{u}) - \bar{v}_i g_i(\bar{u})) + c(\bar{x},\bar{u})\sum_{j=1}^{m} \bar{\mu}_j h_j(\bar{u}), \tag{14}$$

By the hypothesis (i) and (ii), we have:

$$b_i(\bar{x},\bar{u})(f_i(\bar{x}) - f_i(\bar{u})) \geqq \\ \langle \alpha(\bar{x},\bar{u})\xi_i, \eta(\bar{x},\bar{u})\rangle + \rho_i\|\theta(\bar{x},\bar{u})\|^2, \forall \xi_i \in \partial f_i(\bar{u}), \tag{15}$$

$$b_i(\bar{x},\bar{u})(-g_i(\bar{x}) + g_i(\bar{u})) \geqq \\ \langle \alpha(\bar{x},\bar{u})(-\zeta_i), \eta(\bar{x},\bar{u})\rangle + \bar{\rho}_i\|\theta(\bar{x},\bar{u})\|^2, \forall \zeta_i \in \partial g_i(\bar{u}), \tag{16}$$

$$c(\bar{x},\bar{u})(h_j(\bar{x}) - h_j(\bar{u})) \geqq \\ \langle \alpha(\bar{x},\bar{u})\tau_j, \eta(\bar{x},\bar{u})\rangle + \gamma_j\|\theta(\bar{x},\bar{u})\|^2, \forall \tau_j \in \partial h_j(\bar{u}). \tag{17}$$

Multiplying Equation (15) with $\bar{\lambda}_i$ and Equation (16) with $\bar{\lambda}_i \bar{v}_i$, then summing up these equations, we obtain:

$$\begin{aligned}&\sum_{i=1}^{p} b_i(\bar{x},\bar{u})\bar{\lambda}_i(f_i(\bar{x})-f_i(\bar{u})-\bar{v}_i g_i(\bar{x})+\bar{v}_i g_i(\bar{u}))+\\ &c(\bar{x},\bar{u})\sum_{j=1}^{m}\bar{\mu}_j\left[h_j(\bar{x})-h_j(\bar{u})\right]\geqq\\ &\left\langle \alpha(\bar{x},\bar{u})[\sum_{i=1}^{p}\bar{\lambda}_i(\xi_i-\bar{v}_i\zeta_i)+\sum_{j=1}^{m}\bar{\mu}_j\tau_j],\eta(\bar{x},\bar{u})\right\rangle+\\ &\left[\sum_{i=1}^{p}\bar{\lambda}_i(\rho_i+\bar{v}_i\bar{\rho}_i)+\sum_{j=1}^{m}\bar{\mu}_j\gamma_j\right]\left\|\theta(\bar{x},\bar{u})\right\|^2.\end{aligned} \tag{18}$$

According to the constraint condition of (MFD_1) and the hypothesis (iv), we can conclude that

$$\begin{aligned}&\sum_{i=1}^{p} b_i(\bar{x},\bar{u})\bar{\lambda}_i(f_i(\bar{x})-f_i(\bar{u})-\bar{v}_i g_i(\bar{x})+\bar{v}_i g_i(\bar{u}))+\\ &c(\bar{x},\bar{u})\sum_{j=1}^{m}\bar{\mu}_j[h_j(\bar{x})-h_j(\bar{u})]\geqq 0.\end{aligned} \tag{19}$$

We have a contradiction. Hence, the result is true.

Theorem 2. (Weak Duality) Let $\bar{x}\in X^0$ and $(\bar{u},\bar{\lambda},\bar{\mu},\bar{v})\in D^0$. Suppose that:

(i) $f_i-\bar{v}_i g_i$ is $(b_i,\alpha)-\rho_i-(\eta,\theta)$ – strictly quasi-invex and regular at $\bar{u}, i=1,2,\cdots,p$;

(ii) h_j is $(c,\alpha)-\gamma_j-(\eta,\theta)$ – invex at $\bar{u}$, $j=1,2,\cdots,m$;

(iii) $b_i(\bar{x},\bar{u})\geqq 0, i=1,2,\cdots,p, c(\bar{x},\bar{u})>0$;

(iv) $\sum_{i=1}^{p}\bar{\lambda}_i\rho_i+\sum_{j=1}^{m}\bar{\mu}_j\gamma_j\geqq 0$;

Then $F(\bar{x})\nleq\bar{v}$.

Proof: Suppose contrary to the result that $F(\bar{x})\leq\bar{v}$. That implies:

$$\frac{f_i(\bar{x})}{g_i(\bar{x})}\leqq\bar{v}_i,\text{for } i=1,2,\cdots,p, \tag{20}$$

and:

$$\frac{f_k(\bar{x})}{g_k(\bar{x})}<\bar{v}_k,\text{for some } k\in\{1,2,\cdots,p\}. \tag{21}$$

That is:

$$f_i(\bar{x})-\bar{v}_i g_i(\bar{x})\leqq f_i(\bar{u})-\bar{v}_i g_i(\bar{u}),\text{for } i=1,2,\cdots,p, \tag{22}$$

and:

$$f_k(\bar{x})-\bar{v}_k g_k(\bar{x})<f_k(\bar{u})-\bar{v}_k g_k(\bar{u}),\text{for some } k\in\{1,2,\cdots,p\}. \tag{23}$$

Since $b_i(\bar{x},\bar{u})\geqq 0$, the above inequalities yield:

$$b_i(\bar{x},\bar{u})[(f_i(\bar{x})-\bar{v}_i g_i(\bar{x}))-(f_i(\bar{u})-\bar{v}_i g_i(\bar{u}))]\geqq 0. \tag{24}$$

By the hypothesis (i), we have:

$$\begin{aligned}&\left\langle\alpha(\bar{x},\bar{u})(\xi_i-\bar{v}_i\zeta_i),\eta(\bar{x},\bar{u})\right\rangle+\rho_i\left\|\theta(\bar{x},\bar{u})\right\|^2<0,\\ &\forall\xi_i\in\partial f_i(\bar{u}),\zeta_i\in\partial g_i(\bar{u}),i=1,2,\cdots,p.\end{aligned} \tag{25}$$

Since $\bar{\lambda}_i>0$, we obtain:

$$\left\langle\alpha(\bar{x},\bar{u})\sum_{i=1}^{p}\bar{\lambda}_i(\xi_i-\bar{v}_i\zeta_i),\eta(\bar{x},\bar{u})\right\rangle+\sum_{i=1}^{p}\bar{\lambda}_i\rho_i\left\|\theta(\bar{x},\bar{u})\right\|^2<0. \tag{26}$$

According to the hypothesis (ii), we get:

$$\begin{aligned}&c(\bar{x},\bar{u})(h_j(\bar{x})-h_j(\bar{u}))\geqq\left\langle\alpha(\bar{x},\bar{u})\tau_j,\eta(\bar{x},\bar{u})\right\rangle+\\ &\gamma_j\left\|\theta(\bar{x},\bar{u})\right\|^2,\forall\tau_j\in\partial h_j(\bar{u}), j=1,2,\cdots,m.\end{aligned} \tag{27}$$

Since $\bar{x}\in X^0$ and $(\bar{u},\bar{\lambda},\bar{\mu},\bar{v})\in D^0$, it follows that:

$$\begin{aligned}&c(\bar{x},\bar{u})(\sum_{j=1}^{m}\bar{\mu}_j h_j(\bar{x})-\sum_{j=1}^{m}\bar{\mu}_j h_j(\bar{u}))\geqq\\ &\left\langle\alpha(\bar{x},\bar{u})\sum_{j=1}^{m}\bar{\mu}_j\tau_j,\eta(\bar{x},\bar{u})\right\rangle+\sum_{j=1}^{m}\bar{\mu}_j\gamma_j\left\|\theta(\bar{x},\bar{u})\right\|^2\geqq 0.\end{aligned} \tag{28}$$

We have:

$$\begin{aligned}&\left\langle\alpha(\bar{x},\bar{u})\left[\sum_{i=1}^{p}\bar{\lambda}_i(\xi_i-\bar{v}_i\zeta_i)+\sum_{j=1}^{m}\bar{\mu}_j\tau_j\right],\eta(\bar{x},\bar{u})\right\rangle+\\ &\left[\sum_{i=1}^{p}\bar{\lambda}_i\rho_i+\sum_{j=1}^{m}\bar{\mu}_j\gamma_j\right]\left\|\theta(\bar{x},\bar{u})\right\|^2<0.\end{aligned} \tag{29}$$

By the hypothesis (i) and (ii), we have:

$$\begin{aligned}&b_i(\bar{x},\bar{u})(f_i(\bar{x})-f_i(\bar{u}))\geqq\left\langle\alpha(\bar{x},\bar{u})\xi_i,\eta(\bar{x},\bar{u})\right\rangle+\\ &\rho_i\left\|\theta(\bar{x},\bar{u})\right\|^2,\forall\xi_i\in\partial f_i(\bar{u}),\end{aligned} \tag{30}$$

$$\begin{aligned}&b_i(\bar{x},\bar{u})(-g_i(\bar{x})+g_i(\bar{u}))\geqq\left\langle\alpha(\bar{x},\bar{u})(-\zeta_i),\eta(\bar{x},\bar{u})\right\rangle,\\ &\bar{\rho}_i\left\|\theta(\bar{x},\bar{u})\right\|^2,\forall\zeta_i\in\partial g_i(\bar{u}),\end{aligned} \tag{31}$$

$$\begin{aligned}&c(\bar{x},\bar{u})(h_j(\bar{x})-h_j(\bar{u}))\geqq\left\langle\alpha(\bar{x},\bar{u})\tau_j,\eta(\bar{x},\bar{u})\right\rangle+\\ &\gamma_j\left\|\theta(\bar{x},\bar{u})\right\|^2,\forall\tau_j\in\partial h_j(\bar{u}).\end{aligned} \tag{32}$$

Using the constraint condition of (MFD_1), we obtain:

$$\sum_{i=1}^{p}\bar{\lambda}_i\rho_i+\sum_{j=1}^{m}\bar{\mu}_j\gamma_j<0. \tag{33}$$

We have a contradiction to hypothesis (iv). Hence, the proof is complete.

Theorem 3 (Strong Duality). Let $\bar{x}$ be an efficient solution of (MFP) at which the generalized Kuhn-Tucker constraint qualification is satisfied. Then there exists $\bar{\lambda}\in\Lambda^{++},\bar{\mu}\in R_+^m$ and $\bar{v}\in R_+^p$, such that $(\bar{x},\bar{\lambda},\bar{\mu},\bar{v})$ is a feasible solution of (MFD_1), and the values of the objective functions for (MFP) and (MFD$_1$) are equal at $\bar{x}$ and $(\bar{x},\bar{\lambda},\bar{\mu},\bar{v})$.

Furthermore, if the assumptions of Theorem 4 or Theorem 5 hold for all feasible solutions of (MFP) and (MFD$_1$), then $(\bar{x},\bar{\lambda},\bar{\mu},\bar{v})$ is an efficient solution of (MFD$_1$).

Proof: Since is an efficient solution of (MFP) for which the generalized Kuhn-Tucker constraint qualification is satisfied, then there exists $\bar{\lambda}\in\Lambda^{++},\bar{\mu}\in R_+^m$ and $\bar{v}\in R_+^p$, such that $(\bar{x},\bar{\lambda},\bar{\mu},\bar{v})$ satisfies the conditions of (GKT). The conditions imply that $\bar{v}_i=\dfrac{f_i(\bar{x})}{g_i(\bar{x})}$, for all $i=1,2,\cdots,p$.

It is clear that the values of the objective functions for (MFP) and (MFD$_1$) are equal at $\bar{x}$ and $(\bar{x},\bar{\lambda},\bar{\mu},\bar{v})$.

In addition, from the weak duality Theorem 4 or Theorem 5, for any feasible solution $(x,\lambda,\mu,v)\in D^0$, the following cannot hold:

$$\frac{f_i(\bar{x})}{g_i(\bar{x})}\leqq v_i,\text{for } i=1,2,\cdots,p, \tag{34}$$

and:

$$\frac{f_k(\bar{x})}{g_k(\bar{x})}<v_k,\text{for some } k\in\{1,2,\cdots,p\}. \tag{35}$$

Hence, we conclude that $(\bar{u},\bar{\lambda},\bar{\mu},\bar{v})$ is an efficient solution of (MFD$_1$).

Theorem 4 (Strict Converse Duality): let $\bar{x}$ and $(\bar{u},\bar{\lambda},\bar{\mu},\bar{v})$ be an efficient solution for (MFP) and (MFD$_1$), respectively with $\bar{v}_i=\dfrac{f_i(\bar{x})}{g_i(\bar{x})}$, for all $i=1,2,\cdots,p$.

Suppose that:

(i) $f_i-\bar{v}_i g_i$ is $(b_i,\alpha)-\rho_i-(\eta,\theta)$ – strictly invex and regular at $\bar{u}, i=1,2,\cdots,p$;

(ii) h_j is $(c,\alpha)-\gamma_j-(\eta,\theta)$ – invex at $\bar{u}$, $j=1,2,\cdots,m$;

(iii) $b_i(\bar{x},\bar{u})\geqq 0(i=1,2,\cdots,p),c(\bar{x},\bar{u})\geqq 0$;

(iv) $\sum_{i=1}^{p}\bar{\lambda}_i\rho_i+\sum_{j=1}^{m}\bar{\mu}_j\gamma_j\geqq 0$.

Then $\bar{x}=\bar{u}$, $\bar{u}$ is an efficient solution for (MFP).

Proof: Suppose contrary to the result that $\bar{x}\neq\bar{u}$.

By the hypothesis (i), we have:

$$\begin{aligned}&b_i(\bar{x},\bar{u})[(f_i(\bar{x})-\bar{v}_i g_i(\bar{x}))-(f_i(\bar{u})-\bar{v}_i g_i(\bar{u}))]>\\&\left\langle\alpha(\bar{x},\bar{u})(\xi_i-\bar{v}_i\zeta_i),\eta(\bar{x},\bar{u})\right\rangle+\rho_i\left\|\theta(\bar{x},\bar{u})\right\|^2,\\&\forall\xi_i\in\partial f_i(\bar{u}),\zeta_i\in\partial g_i(\bar{u}),i=1,2,\cdots,p.\end{aligned} \tag{36}$$

From the constraint condition of (MFD$_1$) and $b_i(\bar{x},\bar{u})\geqq 0$, we get:

$$\left\langle\alpha(\bar{x},\bar{u})(\xi_i-\bar{v}_i\zeta_i),\eta(\bar{x},\bar{u})\right\rangle+\rho_i\left\|\theta(\bar{x},\bar{u})\right\|^2<0. \tag{37}$$

Since $\bar{\lambda}_i>0$, the above inequality yields:

$$\begin{aligned}&\left\langle\alpha(\bar{x},\bar{u})\sum_{i=1}^{p}\bar{\lambda}_i(\xi_i-\bar{v}_i\zeta_i),\eta(\bar{x},\bar{u})\right\rangle+\\&\sum_{i=1}^{p}\bar{\lambda}_i\rho_i\left\|\theta(\bar{x},\bar{u})\right\|^2<0\end{aligned}. \tag{38}$$

Using the hypothesis (ii), we have:

$$\begin{aligned}&c(\bar{x},\bar{u})(h_j(\bar{x})-h_j(\bar{u}))\geqq\left\langle\alpha(\bar{x},\bar{u})\tau_j,\eta(\bar{x},\bar{u})\right\rangle+\\&\gamma_j\left\|\theta(\bar{x},\bar{u})\right\|^2,\forall\tau_j\in\partial h_j(\bar{u}),j=1,2,\cdots,m,\end{aligned} \tag{39}$$

Since $\bar{\mu}_j\geqq 0, j=1,2,\cdots,m$, the inequality follows:

$$\begin{aligned}&c(\bar{x},\bar{u})(\sum_{j=1}^{m}\bar{\mu}_j h_j(\bar{x})-\sum_{j=1}^{m}\bar{\mu}_j h_j(\bar{u}))\geqq\\&\left\langle\alpha(\bar{x},\bar{u})\sum_{j=1}^{m}\bar{\mu}_j\tau_j,\eta(\bar{x},\bar{u})\right\rangle+\sum_{j=1}^{m}\bar{\mu}_j\gamma_j\left\|\theta(\bar{x},\bar{u})\right\|^2\geqq 0,\end{aligned} \tag{40}$$

From the constraint condition of (MFD$_1$) and the above inequality, we obtain

$$\left\langle\alpha(\bar{x},\bar{u})\sum_{j=1}^{m}\bar{\mu}_j\tau_j,\eta(\bar{x},\bar{u})\right\rangle+\sum_{j=1}^{m}\bar{\mu}_j\gamma_j\left\|\theta(\bar{x},\bar{u})\right\|^2\leqq 0, \tag{41}$$

Summing up Equations (38) and (41), we concludes that

$$\begin{aligned}&\left\langle\alpha(\bar{x},\bar{u})\left[\sum_{i=1}^{p}\bar{\lambda}_i(\xi_i-\bar{v}_i\zeta_i)+\sum_{j=1}^{m}\bar{\mu}_j\tau_j\right],\eta(\bar{x},\bar{u})\right\rangle+\\&\left[\sum_{i=1}^{p}\bar{\lambda}_i\rho_i+\sum_{j=1}^{m}\bar{\mu}_j\gamma_j\right]\left\|\theta(\bar{x},\bar{u})\right\|^2<0\end{aligned}. \tag{42}$$

The above inequality together with the hypothesis (iv) implies

$$\left\langle\alpha(\bar{x},\bar{u})\left[\sum_{i=1}^{p}\bar{\lambda}_i(\xi_i-\bar{v}_i\zeta_i)+\sum_{i=1}^{p}\bar{\mu}_j\tau_j\right],\eta(\bar{x},\bar{u})\right\rangle<0. \tag{43}$$

We have a contradiction. Hence $\bar{x}=\bar{u}$.

4 Wolfe duality

In this section, we consider the Wolfe type dual model for (MFP). The Wolfe type dual can be formulated as follows: (MFD$_2$):

$$\max G(u,\mu)=\left(\frac{f_1(u)+\sum_{j=1}^{m}\mu_j h_j(u)}{g_1(u)},\cdots,\frac{f_p(u)+\sum_{j=1}^{m}\mu_j h_j(u)}{g_p(u)}\right)$$
$$s.t.\ \ 0\in\sum_{i=1}^{p}\lambda_i g_i(u)[\partial f_i(u)+\sum_{j=1}^{m}\mu_j\partial h_j(u)]$$
$$-\sum_{i=1}^{p}\lambda_i[f_i(u)+\sum_{j=1}^{m}\mu_j h_j(u)]\partial g_i(u),$$
$$\lambda\in\Lambda^{++},\mu\in R_+^m,u\in X.$$

Let:

$$E^0=\{(u,\lambda,\mu)\in X\times\Lambda^{++}\times R_+^m$$
$$\Big|\,0\in\sum_{i=1}^{p}\lambda_i g_i(u)[\partial f_i(u)-v_i\partial g_i(u)]\sum_{j=1}^{m}\mu_j\partial h_j(u),$$
$$f_i(u)-v_i g_i(u)\geqq 0 \text{ with } i=1,2,\cdots,p,\sum_{j=1}^{m}\mu_j h_j(u)\geqq 0\}$$

denote the feasible set of (MFD_2).

Theorem 5. (Weak Duality)

Let $\bar{x}\in X^0$ and $(\bar{u},\bar{\lambda},\bar{\mu})\in E^0$. Suppose that:

(i) f_i is $(b_i,\alpha)-\rho_i-(\eta,\theta)$ – invex at $\bar{u}$, $-g_i$ is $(b_i,\alpha)-\bar{\rho}_i-(\eta,\theta)$ – invex and regular at $\bar{u},i=1,2,\cdots,p$;

(ii) h_j is $(c,\alpha)-\gamma_j-(\eta,\theta)$ – invex at $\bar{u}$, $j=1,2,\cdots,m$;

(iii) $b_i(\bar{x},\bar{u})>0,i=1,2,\cdots,p,0<c(\bar{x},\bar{u})\leqq 1$;

(iv) $\rho_i+b_i(\bar{x},\bar{u})\sum_{j=1}^{m}\bar{\mu}_j\gamma_j\geqq 0,\bar{\rho}_i\geqq 0,i=1,2,\cdots,p$.

Then $F(\bar{x})\nleq G(\bar{u},\bar{\mu})$.

Proof: since $(\bar{u},\bar{\lambda},\bar{\mu},)\in E^0$, it follows that there exists $\xi_i\in\partial f_i(\bar{u}),\zeta_i\in\partial g_i(\bar{u}),\tau_j\in\partial h_j(\bar{u}),i=1,2,\cdots,p,$ $j=1,2,\cdots,m$, such that:

$$\sum_{i=1}^{p}\bar{\lambda}_i g_i(\bar{u})\left[\xi_i+\sum_{j=1}^{m}\bar{\mu}_j\tau_j\right]-\sum_{i=1}^{p}\bar{\lambda}_i[f_i(\bar{u})-\sum_{j=1}^{m}\bar{\mu}_j h_j(\bar{u})]\zeta_i=0$$

Suppose contrary to the result of the theorem that $F(\bar{x})\nleq G(\bar{u},\bar{\mu})$. That implies:

$$\frac{f_i(\bar{x})}{g_i(\bar{x})}\leqq\frac{f_i(\bar{u})+\sum_{j=1}^{m}\bar{\mu}_j h_j(\bar{u})}{g_i(\bar{u})},\text{for } i=1,2,\cdots,p, \quad (44)$$

and

$$\frac{f_k(\bar{x})}{g_k(\bar{x})}<\frac{f_k(\bar{u})+\sum_{j=1}^{m}\bar{\mu}_j h_j(\bar{u})}{g_k(\bar{u})},\text{for some } k\in\{1,2,\cdots,p\}. \quad (45)$$

Therefore:

$$f_i(\bar{x})g_i(\bar{u})-g_i(\bar{x})[f_i(\bar{u})+\sum_{j=1}^{m}\bar{\mu}_j h_j(\bar{u})], \quad (46)$$
$$\leqq 0,\text{for } i=1,2,\cdots,p$$

and:

$$f_k(\bar{x})g_k(\bar{u})-g_k(\bar{x})\left[f_k(\bar{u})+\sum_{j=1}^{m}\bar{\mu}_j h_j(\bar{u})\right]<0 \quad (47)$$
for some $k\in\{1,2,\cdots,p\}$.

Since:

$b_i(\bar{x},\bar{u})>0,g_i(\bar{u})>0$ and $\bar{\mu}_j h_j(\bar{x})\leqq 0,(i=1,2,\cdots,p)$, $(j=1,2,\cdots,m)$, the above inequalities yield

$$\sum_{i=1}^{p}\bar{\lambda}_i g_i(\bar{u})b_i(\bar{x},\bar{u})[f_i(\bar{x})+\sum_{j=1}^{m}\bar{\mu}_j h_j(\bar{x})]-$$
$$\sum_{i=1}^{p}\bar{\lambda}_i g_i(\bar{x})b_i(\bar{x},\bar{u})[f_i(\bar{u})+\sum_{j=1}^{m}\bar{\mu}_j h_j(\bar{u})]<0. \quad (48)$$

Thus:

$$\sum_{i=1}^{p}\bar{\lambda}_i g_i(\bar{u})b_i(\bar{x},\bar{u})\Big[(f_i(\bar{x})-f_i(\bar{u}))+$$
$$\left(\sum_{j=1}^{m}\bar{\mu}_j h_j(\bar{x})-\sum_{j=1}^{m}\bar{\mu}_j h_j(\bar{u})\right)\Big]-\sum_{i=1}^{p}\bar{\lambda}_i\left[f_i(\bar{u})+\sum_{j=1}^{m}\bar{\mu}_j h_j(\bar{u})\right]$$
$$b_i(\bar{x},\bar{u})(g_i(\bar{x})-g_i(\bar{u}))<0. \quad (49)$$

By the hypothesis (i) and (ii), we have:

$$b_i(\bar{x},\bar{u})(f_i(\bar{x})-f_i(\bar{u}))\geqq\left\langle\alpha(\bar{x},\bar{u})\xi_i,\eta(\bar{x},\bar{u})\right\rangle+$$
$$\rho_i\left\|\theta(\bar{x},\bar{u})\right\|^2,\forall\xi_i\in\partial f_i(\bar{u}). \quad (50)$$

$$b_i(\bar{x},\bar{u})(-g_i(\bar{x})+g_i(\bar{u}))\geqq$$
$$\left\langle\alpha(\bar{x},\bar{u})(-\zeta_i),\eta(\bar{x},\bar{u})\right\rangle+\bar{\rho}_i\left\|\theta(\bar{x},\bar{u})\right\|^2,\forall\zeta_i\in\partial g_i(\bar{u}). \quad (51)$$

$$c(\bar{x},\bar{u})(h_j(\bar{x})-h_j(\bar{u}))\geqq$$
$$\left\langle\alpha(\bar{x},\bar{u})\tau_j,\eta(\bar{x},\bar{u})\right\rangle+\gamma_j\left\|\theta(\bar{x},\bar{u})\right\|^2,\forall\tau_j\in\partial h_j(\bar{u}). \quad (52)$$

Since $0<c(\bar{x},\bar{u})\leqq 1$, Equation (52) implies:

$$h_j(\bar{x})-h_j(\bar{u})\geqq\left\langle\alpha(\bar{x},\bar{u})\tau_j,\eta(\bar{x},\bar{u})\right\rangle+\gamma_j\left\|\theta(\bar{x},\bar{u})\right\|^2. \quad (53)$$

Using $\bar{\mu}_j\geqq 0(j=1,2,\cdots,m)$, Equation (53) yields:

$$\sum_{j=1}^{m}\bar{\mu}_j h_j(\bar{x})-\sum_{j=1}^{m}\bar{\mu}_j h_j(\bar{u})\geqq \left\langle \alpha(\bar{x},\bar{u})\sum_{j=1}^{m}\bar{\mu}_j\tau_j,\eta(\bar{x},\bar{u})\right\rangle+\sum_{j=1}^{m}\bar{\mu}_j\gamma_j\left\|\theta(\bar{x},\bar{u})\right\|^2. \tag{54}$$

Multiplying Equations (50) and (54) with $\bar{\lambda}_i g_i(\bar{u})$ and $\bar{\lambda}_i g_i(\bar{u})b_i(\bar{x},\bar{u})$, respectively, and multiplying Equation (51) with $\bar{\lambda}_i(f_i(\bar{u})+\sum_{j=1}^{m}\bar{\mu}_j h_j(\bar{u}))$, then summing these inequalities together, we obtain:

$$\begin{aligned}&\sum_{i=1}^{p}\bar{\lambda}_i g_i(\bar{u})b_i(\bar{x},\bar{u})\Big[\Big(f_i(\bar{x})-f_i(\bar{u})\Big)+\\&\left(\sum_{j=1}^{m}\bar{\mu}_j h_j(\bar{x})-\sum_{j=1}^{m}\bar{\mu}_j h_j(\bar{u})\right)\Big]-\\&\sum_{i=1}^{p}\bar{\lambda}_i\left[f_i(\bar{u})+\sum_{j=1}^{m}\bar{\mu}_j h_j(\bar{u})\right]b_i(\bar{x},\bar{u})(g_i(\bar{x})-g_i(\bar{u}))\geqq\\&\left\langle\alpha(\bar{x},\bar{u})\left[\sum_{i=1}^{p}\bar{\lambda}_i g_i(\bar{u})\left(\xi_i+\sum_{j=1}^{m}\bar{\mu}_j\tau_j\right)-\sum_{i=1}^{p}\bar{\lambda}_i(f_i(\bar{u})+\right.\right.\\&\left.\left.\sum_{j=1}^{m}\bar{\mu}_j h_j(\bar{u}))\zeta\right]_i,\eta(\bar{x},\bar{u})\right\rangle+\left[\sum_{i=1}^{p}\bar{\lambda}_i g_i(\bar{u})(\rho_i+b_i(\bar{x},\bar{u})\right.\\&\left.\sum_{j=1}^{m}\bar{\mu}_j\gamma_j)+\bar{\rho}_i(f_i(\bar{u})+\sum_{j=1}^{m}\bar{\mu}_j h_j(\bar{u}))\right]\left\|\theta(\bar{x},\bar{u})\right\|^2.\end{aligned} \tag{55}$$

The above inequality together with the constraint conditions and hypothesis (iv), implies

$$\begin{aligned}&\sum_{i=1}^{p}\bar{\lambda}_i g_i(\bar{u})b_i(\bar{x},\bar{u})\Big[(f_i(\bar{x})-f_i(\bar{u}))+\\&\left(\sum_{j=1}^{m}\bar{\mu}_j h_j(\bar{x})-\sum_{j=1}^{m}\bar{\mu}_j h_j(\bar{u})\right)\Big]-\\&\sum_{i=1}^{p}\bar{\lambda}_i\left[f_i(\bar{u})+\sum_{j=1}^{m}\bar{\mu}_j h_j(\bar{u})\right],\\&b_i(\bar{x},\bar{u})(g_i(\bar{x})-g_i(\bar{u}))\geqq 0,\end{aligned} \tag{56}$$

which contradicts Equation (49). This completes the proof.

Theorem 6: (Weak Duality). Let $\bar{x}\in X^0$ and $(\bar{u},\bar{\lambda},\bar{\mu})\in E^0$. Suppose that:

(i) $f_i(\cdot)+\sum_{j=1}^{m}\bar{\mu}_j h_j(\cdot)$ is $(b_i,\alpha)-\rho_i-(\eta,\theta)-$invex and regular at $\bar{u}$, $-g_i(\cdot)$ is $(b_i,\alpha)-\bar{\rho}_i-(\eta,\theta)-$invex and regular at $\bar{u}, i=1,2,\cdots,p$;

(ii) $b_i(\bar{x},\bar{u})>0, i=1,2,\cdots,p$;

(iii) $\rho_i\geqq 0,\bar{\rho}_i\geqq 0, i=1,2,\cdots,p$.

Then $F(\bar{x})\nleq G(\bar{u},\bar{\mu})$.

Proof: the proof is similar to the Theorem 5.

Theorem 7. (Strong Duality) Let $\bar{x}$ be an efficient solution of (MFP). Suppose that there exists $\bar{\lambda}\in\Lambda^{++}$ and $\bar{\mu}\in R_+^m$, such that $\bar{\mu}_j h_j(\bar{x})=0(j=1,2,\cdots,m)$ and $(\bar{u},\bar{\lambda},\bar{\mu})$ is a feasible solution of (MFD_2). Then the objective function values of (MFP) and (MFD_2) are equal at $\bar{x}$ and $(\bar{u},\bar{\lambda},\bar{\mu})$.

Furthermore, if the assumptions of Theorem 8 or Theorem 9 hold for all feasible solutions of (MFP) and (MFD_2), then $(\bar{u},\bar{\lambda},\bar{\mu})$ is an efficient solution of (MFD_2).

Proof: Based on the weak duality theorem, we can obtain the result that $(\bar{u},\bar{\lambda},\bar{\mu})$ is an efficient solution of (MFD_2).

Theorem 8. (Strict Converse Duality) Let $\bar{x}$ and $(\bar{u},\bar{\lambda},\bar{\mu})$ be an efficient solutions of (MFP) and (MFD_2), respectively. Suppose that:

(i) $\dfrac{f_i(\bar{x})}{g_i(\bar{x})}\leqq\dfrac{f_i(\bar{u})+\sum_{j=1}^{m}\bar{\mu}_j h_j(\bar{u})}{g_i(\bar{u})}$, $i=1,2,\cdots,p$;

(i) $f_i(\cdot)+\sum_{j=1}^{m}\bar{\mu}_j h_j(\cdot)$ is $(b_i,\alpha)-\rho_i-(\eta,\theta)-$ invex and regular at $\bar{u}$, $-g_i(\cdot)$ is $(b_i,\alpha)-\bar{\rho}_i-(\eta,\theta)-$ invex and regular at $\bar{u}, i=1,2,\cdots,p$;

(iii) $b_i(\bar{x},\bar{u})>0,\rho_i\geqq 0,\bar{\rho}_i\geqq 0, i=1,2,\cdots,p$;

Then $\bar{x}=\bar{u}$, that is, $\bar{u}$ is an efficient solution of (MFP).

5 Discussion and conclusion

Throughout this paper, we have established two dual models namely parametric duality models and Wolfe duality models for the multiobjective fractional programming problem. Several duality results were derived and proved with the help of $(b,\alpha)-\rho-(\eta,\theta)-$ Invexity assumption. The results should be further opportunities for exploiting this structure of the multiobjective fractional programming problem.

Acknowledgments

This work is supported by Scientific Research Program Funded by Shaanxi Provincial Education Department (Program No. 12JK0875) and Scientific Research Development Program Funded by Xi'an University of Science and Technology (Program No: 201349).

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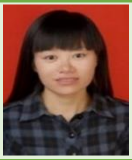

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