A few expanding integrable models of WKI hierarchy and their Hamiltonian structures

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Abstract

The integrable coupling of the WKI hierarchy is obtained by the perturbation approach. With the help of a higher dimensional loop algebra, the coupling integrable couplings of the WKI hierarchy are obtained, respectively. Their Hamiltonian structures are worked out by employing the component-trace identities and variational identity.

Keywords: coupling integrable couplings, component-trace identities, perturbation equation

1 Introduction

The notion on integrable couplings was introduced when study of Virasoro symmetric algebras [1, 2]. To find as many new integrable systems and their integrable couplings as possible and to elucidate in depth their algebraic and geometric properties are of both theoretical and practical value. During the past few years, some interesting integrable couplings and associated properties of some known interesting integrable hierarchies, such as the AKNS hierarchy, the KN hierarchy, the Burger hierarchy, etc. were obtained [3-13] In order to get Hamiltonian structures of integrable couplings, Guo and Zhang proposed the quadratic-form identity [14]. After this, Ma and Chen [15, 16] built the variational identity and generalized the quadratic-form identity and obtained some integrable couplings and their Hamiltonian structures. Recently, Ma and Zhang [17] proposed the notion on component-trace identities. They are very effective to construct the Hamiltonian structure of the perturbation equation. In Ref. [18], Ma and Gao proposed the notion called coupling integrable couplings of the nonlinear Schrödinger equation and associated symmetry properties, etc. Based on this, Zhang and Tam [19] constructed a few higher dimensional Lie algebras to obtain the coupling integrable couplings of the AKNS hierarchy and the KN hierarchy.

In the paper, we first give the first-order perturbation equation of the WKI hierarchy and its Hamiltonian structure is worked out by employing the component-trace identities. Then we use the way presented in Ref. [18-19] to investigate the coupling integrable couplings of the WKI hierarchy. In Refs. [18-19], the author didn't obtain the Hamiltonian structure of the coupling integrable couplings, while in the paper the Hamiltonian structures of the coupling integrable couplings of the WKI hierarchy will be worked out by using the variational identity.

2 The perturbation equation of the WKI hierarchy and its Hamiltonian structure

Yao and Zhang [20] utilized Tu scheme to obtain the multi-component WKI hierarchy. In this section, we take the perturbation way to deduce the integrable coupling of the WKI hierarchy and employ the component-trace identities to generate its Hamiltonian structure.

Consider the isospectral problem of the WKI hierarchy

\[
\phi_s = U \phi_s U^{-1} = \left( \begin{array}{c} -i \lambda \varphi_s \\ \varphi_s \end{array} \right),
\]

(1)

where \( u_1, u_2 \) are potentials and \( \lambda \) is the spectral parameter. By means of constructing a proper time evolution

\[
\phi_s = V^{(s)} \phi_s V^{(s)^{-1}} = \sum_{m=0}^{\infty} \left( \begin{array}{c} \lambda a_m \\ i \lambda u_m a_m \\ -\lambda a_m \end{array} \right) \phi_s
\]

(2)

and using the zero-curvature equation, we have the WKI hierarchy:

\[
\begin{pmatrix}
0 \\
\partial^2 + \frac{u_2}{\sqrt{1-u_2 u_1}} \\
\frac{u_1}{\sqrt{1-u_2 u_1}}
\end{pmatrix} = \begin{pmatrix}
\frac{u_2}{\sqrt{1-u_2 u_1}} \\
\frac{u_1}{\sqrt{1-u_2 u_1}} \\
0
\end{pmatrix},
\]

(3)

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where:

\[
L = \left( \begin{array}{ccc}
\frac{i}{2} \partial_x - \frac{i}{2} \frac{u_1}{\sqrt{1-u_2}} \partial^{-1} - \frac{u_1}{\sqrt{1-u_2}} \partial^{-1} & \frac{i}{2} \frac{u_2}{\sqrt{1-u_2}} \partial^{-1} & -i \\
\frac{i}{2} \frac{u_1}{\sqrt{1-u_2}} \partial^{-1} - \frac{u_1}{\sqrt{1-u_2}} \partial^{-1} & \frac{i}{2} \frac{u_2}{\sqrt{1-u_2}} \partial^{-1} & -i \\
-i & -i & \frac{i}{2} \frac{u_2}{\sqrt{1-u_2}} \partial^{-1} - \frac{u_2}{\sqrt{1-u_2}} \partial^{-1} \\
\end{array} \right).
\]  

(4)

Accordingly, when \( n = 0 \), we can get the WKI equation. By using the race identity, we can obtain the Hamiltonian structure of the WKI hierarchy

\[
u_n = J \frac{\delta H_{n-1}}{\delta u},
\]

(5)

where \( H_{n-1} = \int_{a_n}^{b_n} \alpha_n + u_n \partial_x - u_n c_n dx \).

Next, we will construct an integrable coupling of the WKI hierarchy by the perturbation approach and its Hamiltonian structure by the component-trace identities.

Let us take a matrix Lie algebra \( g \) consisting of the following matrices:

\[
A = \begin{bmatrix}
A_0 & A_1 & \cdots & A_N \\
A_0 & A_1 & \vdots & \vdots \\
& \ddots & \ddots & \vdots \\
& & \ddots & A_N \\
0 & A_1 & \cdots & A_N
\end{bmatrix},
\]

(6)

where \( A_i, 0 \leq i \leq N \), are square matrices of the same order. For convenience, we rewrite an element of the Lie algebra \( g \) as a vector of matrices: \( \alpha = (A_0, A_1, \cdots, A_N) \), where the components \( A_i, 1 \leq i \leq N \), are defined by:

\[
A_i = \frac{1}{i !} \frac{\partial^i}{\partial t^i} \int_{x=0}^{t} A(u(t)), u(t) = u + \sum_{i=1}^{N} \epsilon^i \eta_i, 1 \leq i \leq N.
\]

(7)

The enlarged zero-curvature equation by perturbation

\[
\hat{U}_N - \hat{V}_N + [\hat{U}_N, \hat{V}_N] = 0,
\]

where \( \hat{U}_N = u + \sum_{i=1}^{N} \epsilon^i \eta_i, \hat{V}_N = V + \sum_{i=1}^{N} \epsilon^i \eta_i, 1 \leq i \leq N \) give rise to the perturbation equation of the \( N \)th order:

\[
\dot{\eta}_N = \hat{K}_N(\dot{\eta}_N) = (K^T(u), \frac{1}{N !} \frac{\partial^N}{\partial t^N} K^T(\hat{u}_N)), \cdots
\]

\[
\int_{x=0}^{t} K^T(\hat{u}_N)), \cdots
\]

\[
\frac{1}{N !} \frac{\partial^N}{\partial t^N} K^T(\hat{u}_N))],
\]

where the column vector \( \dot{\eta}_N \) of dependent variables is \( \dot{\eta}_N = (u^T, \eta^T_1, \cdots, \eta^T_N) \). In what follows, we focus on the perturbation equation of the first order. We consider an isospectral problem as follows:

\[
\hat{U}(\alpha, \lambda) = \hat{U}_1 = \begin{bmatrix} U_0 & U_1 \\
0 & U_0 \end{bmatrix}, \dot{V}(\alpha, \lambda) = \dot{V}_1 = \begin{bmatrix} V_0 & V_1 \\
0 & V_0 \end{bmatrix},
\]

(8)

where \( U_0 \) and \( V_0 \) are defined by Equations (1) and (2), \( U_1 \) and \( V_1 \) are showed as follows:

\[
U_1 = \frac{1}{11} \frac{\partial}{\partial \epsilon} \int_{x=0}^{t} U(\hat{u}_1) = \frac{1}{11} \frac{\partial}{\partial \epsilon} \left[ \begin{array}{c}
-u_2 \lambda + \epsilon \lambda u_i \\
\lambda \epsilon u_i
\end{array} \right] = \begin{bmatrix} 0 & \lambda u_i \\
\lambda \epsilon u_i & 0 \end{bmatrix},
\]

(9)

\[
V_1 = \frac{\partial}{\partial \epsilon} \int_{x=0}^{t} V(\hat{u}_1) = \frac{\partial}{\partial \epsilon} \left[ \begin{array}{c}
\lambda u_i + \epsilon \lambda d_a \\
f_i + i \lambda u_i a + i \lambda u_i d
\end{array} \right] = \begin{bmatrix} \lambda d & f_i + i \lambda u_i a + i \lambda u_i d \\
f_i + i \lambda u_i a + i \lambda u_i d & -\lambda d \end{bmatrix},
\]

(10)

where:

\[
B = b_i + i \lambda u_i a + \epsilon (f_i + i \lambda u_i a + i \lambda u_i d),
\]

\[
C = c_i + i \lambda u_i a + \epsilon (g_i + i \lambda u_i a + i \lambda u_i d).
\]

The enlarged stationary zero-curvature equation \( \hat{V}_1 = [\hat{U}, \hat{V}] \) equivalently yields:

\[
\begin{aligned}
\hat{V}_1 &= [U, V], \\
\hat{V}_{1i} &= [U, V_i] + [U_i, V].
\end{aligned}
\]

(11)

A direct calculation leads to:
From the recursion relation in Equation (12), we have a recursive formula for determining $f_n$, $g_n$:

$$
\begin{align*}
\begin{bmatrix}
g_n \\
-f_n
\end{bmatrix} &= \begin{bmatrix}
L_{1n} & L_{2n} & L_{3n} & L_{4n}
\end{bmatrix}
\begin{bmatrix}
c_{n-1} \\
-f_{n-1} \\
-c_{n-1} \\
-b_{n-1}
\end{bmatrix}.
\end{align*}
$$

(13)

where:

$$
\begin{align*}
l_n &= \begin{bmatrix}
f_n \\
g_n
\end{bmatrix}
= \begin{bmatrix}
-2b_{n+1} & 0 & 0 & 0 & -u_n & 0 & 0 & 0 \quad \text{if } n = 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \quad \text{if } n = 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \quad \text{if } n = 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \quad \text{if } n = 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \quad \text{if } n = 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \quad \text{if } n = 5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \quad \text{if } n = 6
\end{bmatrix}
\begin{bmatrix}
\alpha_0 \\
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\alpha_4 \\
\alpha_5 \\
\alpha_6
\end{bmatrix}.
\end{align*}
$$

(14)

Equation (12) is a recursive formula for determining $f_n$, $g_n$.
Then the enlarged zero-curvature equation \( \dot{U}_\lambda = \left[ \dot{V}_{\lambda}^{(1)} + \left[ U, V_{\lambda}^{(1)} \right] \right] = 0 \) yields the hierarchy of the first-order perturbation equation:
\[
\dot{U}_\lambda = (u_1, u_2, u_3, u_4)^T = \begin{bmatrix} 0 \\ J \\ J_1 \end{bmatrix} \begin{bmatrix} a_{n+1} \\ c_{n+1} \end{bmatrix},
\]
where
\[
J_1 = \frac{1}{N!} \frac{\partial}{\partial \lambda} \bigg|_{\lambda=0} J(\dot{u}_1) = \begin{bmatrix} 0 \\ -\partial \lambda \end{bmatrix}.
\] (15)

Hence, we can get the following integrable couplings of Equation (3) as follows:
\[
\begin{align*}
\dot{u}_1 &= \frac{i\alpha_1}{\sqrt{1-u_1u_2}} \\
\dot{u}_2 &= -\frac{i\alpha_2}{\sqrt{1-u_1u_2}} \\
\dot{u}_3 &= -\frac{u_1}{2\sqrt{1-u_1u_2}} - \frac{u_1(u_1u_3 + u_1u_4)}{2(1-u_1u_2)} - \frac{u_3}{\sqrt{1-u_1u_2}} \\
\dot{u}_4 &= \frac{u_2}{2\sqrt{1-u_1u_2}} + \frac{u_1(u_1u_3 + u_1u_4)}{2(1-u_1u_2)},
\end{align*}
\] (16)

In order to generate the Hamiltonian structure of the first-order perturbation equation of the WKI hierarchy, we introduce the following theorem:

Theorem 1 [17] Let \( g \) be a matrix Lie algebra consisting of block matrices defined by Equation (6). For a given spectral matrix \( U = U(u, \lambda) = (U_{ij}, U_{i1}, \ldots, U_{iN}) \in g \), we have the variational identity:
\[
\frac{\delta}{\delta u} \int \sum_{i,j=1}^N \alpha_i \mathrm{tr} \left( \sum_{i,j=1}^N V_i \frac{\partial U_i}{\partial \lambda} \right) dx = \lambda \gamma \frac{\partial}{\partial \lambda} \sum_{i,j=1}^N \alpha_i \mathrm{tr} \left( \sum_{i,j=1}^N V_i \frac{\partial U_i}{\partial u} \right),
\] (17)

where \( V = V(v, \lambda) = (V_{ij}, V_{i1}, \ldots, V_{iN}) \in g \) satisfies the zero-curvature equation, all \( \alpha_i \)'s are arbitrary constants with \( \alpha_i \neq 0 \) and \( \gamma \) is the constant determined by
\[
\gamma = -\frac{1}{2} \frac{d}{d \lambda} \ln |V, V|.
\]
This variational identity (17) is called the component-trace identity. For a general integer \( N \), we have:
\[
\frac{\delta}{\delta u} \int \sum_{i,j=1}^N \alpha_i \mathrm{tr} \left( \sum_{i,j=1}^N V_i \frac{\partial U_i}{\partial \lambda} \right) dx = \lambda \gamma \frac{\partial}{\partial \lambda} \mathrm{tr} \left( \sum_{i,j=1}^N V_i \frac{\partial U_i}{\partial u} \right),
\] (18)

which is called the last-component-trace identity. Then, the generating function of Hamiltonian functions for the perturbation equation of \( N \)-th order is computed as follows:
\[
\dot{H}_N(\dot{u}_N) = \int \sum_{i,j=1}^N V_i \frac{\partial U_i}{\partial \lambda},
\]
where \( \dot{u}_N = (\dot{u}_1, \dot{u}_2, \ldots, \dot{u}_N)^T \). This implies that the last-component-trace identity provides the generation function of Hamiltonian functions for the perturbation equation. Then, basing on the generating function of Hamiltonian functions for the original equation \( H(u) = \int V \frac{\partial U}{\partial \lambda} \), we can get the generating function of Hamiltonian functions for the perturbation equations of \( N \)-th order as follows:
\[
\dot{H}_N(\dot{u}_N) = \frac{1}{N!} \frac{\partial^N}{\partial \lambda^N} \bigg|_{\lambda=0} H(u_N) = \int \sum_{i,j=1}^N V_i \frac{\partial U_i}{\partial \lambda}.
\]

By using the above results, a direct calculation reads
\[
\int \sum_{i,j=1}^N V_i \frac{\partial U_i}{\partial \lambda} = \int \left( V_0 \frac{\partial U_0}{\partial \lambda} + V_1 \frac{\partial U_1}{\partial \lambda} \right) = (-2i\lambda d - 2iu_2 f + 2iu_1 g - 2iu_1 b + 2iu_2 c),
\] (20)

\[
\mathrm{tr} \left( V_0 \frac{\partial U_0}{\partial \lambda} \right) = 2i\lambda g, \quad \mathrm{tr} \left( V_1 \frac{\partial U_1}{\partial \lambda} \right) = -2i\lambda f;
\] (21)

Basing on the last-component-trace identities, we have
\[
\frac{\delta}{\delta u} \int \left( -2id_{u_1} - 2iu_2 f_{u_1} + 2iu_1 g_{u_1} - 2iu_1 b_{u_1} + 2iu_2 c_{u_1} \right) dx = 2i(2 + \gamma - n) \left[ g_{u_1} f_n - f_{u_1} c_n - b_{u_1} \right].
\] (22)

Take \( n = 2 \) in above equation gives \( \gamma = -1 \). Thus, the Hamiltonian structure of the perturbation equation of the WKI hierarchy is derived as follows:
\[
\dot{U}_\lambda = (u_1, u_2, u_3, u_4)^T = \begin{bmatrix} 0 \\ J \\ J_1 \end{bmatrix} \frac{\delta H_{N+1}}{\delta u},
\] (23)

where:
\[
H_{N+1} = \frac{1}{n+1} \int (d_{u_1} + u_2 f_{u_1} - u_1 g_{u_1} + u_2 b_{u_1} - u_1 c_{u_1}) dx.
\]
3 The coupling integrable couplings of the WKI hierarchy and its Hamiltonian structure

The coupling of the WKI hierarchy is given in the above. In the section, we will construct the coupling integrable couplings by following the way in Ref. [18], which is introduced as follows.

Given two integrable couplings of the integrable equation \( u_i = K(u) \):

\[
\overline{u}_i = \overline{K}_i(u_i) = \begin{bmatrix} K(u) \\ S(u,v) \end{bmatrix},
\]

\[
\overline{u}_j = \overline{K}_j(u_j) = \begin{bmatrix} K(u) \\ T(u,v) \end{bmatrix},
\]

(24)

(25)

It is verified that \( R^0 \) is a Lie algebra if equipped with Equation (27). Take a basis of \( R^0 \) as follows:

\[
e_i = (e_{ij}, \cdots, e_{aj})^T, e_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases},
\]

(28)

A loop algebra \( \tilde{R}^0 \) corresponding to the Lie algebra \( R^0 \) is defined as:

\[
U = -ie_i(1 + u_i e_i(1) + u_j e_j(1) + u_i e_j(1) + u_j e_i(1)),
\]

\[
V = \sum_{m=1}^{n} \left[ a_m e_i(1 - m) + i u_m a_m e_i(1 - m) + b_m e_j(1 - m) + c_m e_j(1 - m) \right]
\]

\[= d_{mn} e_i(1 - m) + f_{mn} e_j(1 - m) + g_{mn} e_k(1 - m) + i(u_m a_m + u_j d_m) e_i(1 - m) + i(u_j a_m + u_i d_m) e_j(1 - m) + i(u_m a_m + u_i b_m + u_j c_m) e_j(1 - m) + \]

\[= i(u_m a_m + u_j b_m + u_i c_m) e_j(1 - m)\]

A solution to \( V_i = [U, V] \) exhibits that:

\[
a_{mn} = u_m c_{mn} - u_j b_{mn},
\]

\[
i(u_m a_{m+1})_s + b_{mn} = -2ib_{m+1},
\]

\[
i(u_j a_{m+1})_s + c_{mn} = 2ic_{m+1},
\]

\[
d_{mn} = u_j g_{mn} - u_j f_{mn} + u_i c_{mn} - u_j b_{mn},
\]

\[
i(u_j a_{m+1})_s + f_{mn} = -2if_{m+1},
\]

\[i(u_m a_{m+1})_s + u_j d_{m+1} = 2ig_{m+1},
\]

\[
h_{mn} = u_j g_{mn} - u_j p_{mn} + u_i c_{mn} - u_j b_{mn} + u_j q_{mn} - u_j p_{mn},
\]

\[i(u_j a_{m+1})_s + u_j h_{m+1} + u_j h_{m+1} + p_{mn} = -2ip_{m+1},
\]

\[i(u_j a_{m+1})_s + u_j h_{m+1} + u_j h_{m+1} + p_{mn} = 2iq_{m+1},
\]

\[\text{Note:}
\]

A new bigger system is formed as follows:

\[
\tilde{u}_i = \tilde{K}(\tilde{u}) = \begin{bmatrix} K(u) \\ S(u,v) \end{bmatrix}, \tilde{u} = \begin{bmatrix} u \\ v \end{bmatrix}.
\]

(26)

We call Equation (26) coupling integrable couplings of Equations (24) and (25).

First, we will construct a 9-dimensional vector-Lie algebra and its corresponding loop algebra. Consider a vector space [18]:

\[R^0 = \text{span}\{a = (a_1, \cdots, a_9)^T, a_i \in R, i = 1, 2, \cdots, 9\}.
\]

For \( \forall a = (a_1, \cdots, a_9)^T, b = (b_1, \cdots, b_9)^T \), define a commutation operation:

\[\{a, b\} = (a_1 b_1 - a_2 b_2, 2(a_1 b_2 - a_2 b_1), a_1 b_3 - a_2 b_1 + a_3 b_1, a_1 b_3 + a_2 b_1 - a_3 b_1 - a_1 b_3).
\]

(27)

By employing the loop algebra \( \tilde{R}^0 \), we consider the following Lax pair:

\[\tilde{R}^0 = \text{span}\{e_i(n), e_i(n) = e_i \lambda^n, [e_i, e_j] \lambda^{n-m}, 1 \leq i, j \leq 9, m, n \in Z\}.
\]

(29)
\( V^{(n)} = \sum_{k=0}^{n} \left[ a_k e_k(n+1-m) + i u_k a_k e_k(n+1-m) + b_k e_k(n+1-m) \right] + \\
\sum_{k=0}^{n} \left[ c_k e_k(n+1-m) + d_k e_k(n+1-m) + f_k e_k(n+1-m) + g_k e_k(n+1-m) + \\
i u_k a_k e_k(n+1-m) + u_k d_k e_k(n+1-m) + c_k e_k(n+1-m) + i u_k a_k e_k(n+1-m) + \\
qu_k e_k(n+1-m) + i u_k a_k e_k(n+1-m) + u_k d_k e_k(n+1-m) \right]. \)

Therefore, the zero-curvature equation

\[
U_i - V_{z}^{(n)} + [U, V^{(n)}] = 0,
\]

admits the following bigger integrable system:

\[
U_i = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{pmatrix}
\]

\[
= \begin{pmatrix} 0 & 0 & 0 & -\vec{e}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\vec{e}^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\vec{e}^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \end{pmatrix} \begin{pmatrix} c_{n+1} + g_{n+1} + q_{n+1} \\ -b_{n+1} - f_{n+1} - p_{n+1} \\ c_{n+1} \\ -b_{n+1} \\ c_{n+1} + q_{n+1} \\ -b_{n+1} - p_{n+1} \end{pmatrix}
\]

Substituting \( u_k = u_k = 0 \) in Equation (33) reduces to the Equation (14), which is an integrable coupling of the WKI hierarchy; when taking \( u_k = u_k = 0 \) in Equation (33) reduces to another integrable coupling of the WKI hierarchy. So we call Equation (33) the coupling integrable couplings of the WKI hierarchy.

In order to deduce the Hamiltonian structure of Equation (33), we rewrite Equation (30) as follows:

\[
[a, b] = a^T R(b),
\]

where:

\[
R(b) = \begin{pmatrix} 0 & 2b_2 & -2b_1 & 0 & 2b_3 & -2b_2 & 0 & 2b_4 & -2b_3 & 0 \\ b_2 & -2b_2 & 0 & -2b_3 & 0 & b_3 & -2b_2 & 0 & b_4 & 0 \\ -b_2 & 0 & 2b_1 & -b_2 & 0 & 2b_2 & -b_3 & 0 & 2b_1 & 0 \\ 0 & 0 & 0 & 0 & 2b_2 & -2b_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_1 & -2b_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -b_2 & 0 & 2b_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2(b_2 + b_3) & -2(b_1 + b_4) & 0 \\ 0 & 0 & 0 & 0 & 0 & b_3 + b_4 & -2(b_1 + b_2) & 0 & 2(b_1 + b_2) & 0 \\ 0 & 0 & 0 & 0 & 0 & -b_2 & 0 & 0 & 2(b_1 + b_2) & 0 \\ \end{pmatrix}.
\]

Solving the matrix equation for the constant matrix \( F \):

\[
R(b)F = -(R(b)F)^T \text{ and } F^T = F.
\]
Then in terms of $F$, define a linear functional:

$\{a, b\} = a^T F b$. Rewrite the Lax pair as follows:

\[
\begin{bmatrix}
2 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

(35)

By using the above linear functional, we have:

$U = (-i\lambda, \lambda u_1, \lambda u_2, 0, \lambda u_3, \lambda u_4, 0, \lambda u_5, \lambda u_6)^T$,

$V = (\lambda a, b, + i\lambda u_1 a, c, + i\lambda u_2 a, \lambda d, f_1 + i\lambda (u_1 a + u_1 d), g_1 + i\lambda (u_1 a + u_1 d), \lambda h$,

$p_1 + i\lambda (u_1 a + u_1 h + u_1 h), q_1 + i\lambda (u_1 a + u_1 h + u_1 h))^T$.

According the variational identity, we have

$\frac{\partial}{\partial u} \int (-2i(a + d + h) + 2iu_1(c + g + q) - 2iu_2(b + f + p) + 2iu_3c - 2iu_4b + 2iu_5(c + q) - 2iu_6(b + p) dx =$

$\lambda \frac{\partial}{\partial u} \int (2i\lambda(c + g + q), -2i\lambda(b + f + p), 2i\lambda c, -2i\lambda b, 2i\lambda(c + q), -2i\lambda(b + p))^T$.

Comparing the coefficients of $\lambda^{-n+1}$ yields

$\frac{\partial}{\partial u} \int (-2i(a_{n-1} + d_{n-1} + h_{n-1}) + 2iu_1(c_{n-1} + g_{n-1} + q_{n-1}) - 2iu_2(b_{n-1} + f_{n-1} + p_{n-1}) + 2iu_3c_{n-1} -$

$2iu_4b_{n-1} + 2iu_5(c_{n-1} + g_{n-1}) - 2iu_6(b_{n-1} + p_{n-1})) dx = 2i(2 + n - n) +$

$\begin{cases}
-c_{n-1} - g_{n-1} \\
-b_{n-1} + f_{n-1} \\
-c_{n-1} \\
b_{n-1}
\end{cases}$

Taking $n = 2$ in above equation gives $\gamma = -1$.

Hence, the coupling integrable couplings of WKI hierarchy Equation (33) can be written as a Hamiltonian form:
\[ U_1 = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -\partial^2 & 0 & 0 \\ 0 & 0 & \partial^2 & 0 & 0 & 0 \\ 0 & -\partial^2 & 0 & 0 & 0 & -\partial^2 \\ \partial^2 & 0 & 0 & \partial^2 & 0 & 0 \\ 0 & 0 & 0 & -\partial^2 & 0 & -\partial^2 \\ 0 & 0 & \partial^2 & 0 & \partial^2 & 0 \end{pmatrix} \begin{pmatrix} c_{n-1} + g_{n-1} + q_{n-1} \\ -b_{n-1} - f_{n-1} - p_{n-1} \\ c_{n-1} \\ -b_{n-1} \\ c_{n-1} + q_{n-1} \\ -b_{n-1} - p_{n-1} \end{pmatrix} = J \frac{\delta H_{n-1}}{\delta u}, \tag{37} \]

where

\[ H_{n-1} = \frac{1}{n-1} \int ((a_{n-1} + d_{n-1} + h_{n-1}) - u_1(c_{n-1} + g_{n-1} + q_{n-1}) + u_3(b_{n-1} + f_{n-1} + p_{n-1})) \]

\[ u_1c_{n-1} + u_1b_{n-1} - u_3(c_{n-1} + g_{n-1}) + u_5(b_{n-1} + p_{n-1}) dx, \]

\( J \) is a Hamiltonian operator.

4 Conclusions

In Ref. [19], the coupling integrable couplings of KN and AKNS hierarchy are obtained, but their Hamiltonian structures are not given. In the paper, however, the integrable coupling of WKI hierarchy is obtained by the perturbation approach and its Hamiltonian structure is given by using the component-trace identities. Meanwhile, basing on a 9-dimensional Lie algebra, we discuss the coupling integrable couplings of the WKI hierarchy and obtain its Hamiltonian structure by the variational identity.

In the future, we will discuss the perturbation equation and coupling integrable couplings of other hierarchies and their Hamiltonian structures.

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